L - Fuzzy Almost Ideals

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Abstract - *We consider L-Fuzzy Sets where the membership values do not necessarily form a chain. In this general setting we introduce the concept of L-Fuzzy Almost Ideals. So far L-Fuzzy Ideals have been well studied in the case where L is a distributive lattice. In this paper, we show that the natural extension of concept of L-Fuzzy Ideals to the case when L is not necessarily distributive is that of L- Fuzzy Almost Ideals.*

Keywords - *L-Fuzzy Sets, Fuzzy Ideals, L-Fuzzy Ideals, L-Fuzzy Almost Ideals, Non-Distributive lattice, Level set and Almost Level Set.*

I. INTRODUCTION

Zadeh [11] introduced the concept of fuzzy sets in 1965.The theory of L-Fuzzy sets was initiated by J.A.Goguen [1]. In [10] Wang-Jin Liu studied fuzzy subgroups and fuzzy ideals in 1982. In [5], T.K.Mukherjee and M.K.Sen have discussed fuzzy ideals of ring and characterized regular rings by fuzzy ideals and have determined all fuzzy prime ideals of the ring Z of integers. U.M.Swamy and K.L.N.Swamy[9] have studied L-Fuzzy prime ideals and L-Fuzzy maximal ideals, where L is a complete lattice satisfying the infinite meet distributive law. In [12], Zhang Yue has discussed the case of L-Fuzzy sets, where L is a completely distributive lattice. Also he has introduced the concept of a primary L-Fuzzy ideal and a primary L-fuzzy ideal belonging to a prime L-fuzzy ideal and proved some fundamental propositions. M.M.Zahedi [13] has introduced the concepts of a L-fuzzy prime, a L-fuzzy completely prime ideal and some fundamental lemmas are proved. Also, a characterization of a L-fuzzy prime ideal of a ring is given. The concept and some properties of fuzzy ideals on a lattice where introduced by Lehmke.S and he has defined new operations over L-fuzzy sets in[3]. Mohammed M. Atallah [6] has proved the L-fuzzy prime ideal theorem on distributive lattice. In[2], B.B.N.Koguep, G.Nkuimi and C.Lelevarro have discussed the notion of fuzzy prime ideal and have highlighted the difference between fuzzy prime ideal and prime fuzzy ideal. Using the characterization of fuzzy ideal induced by a fuzzy set and also they have showed that a fuzzy ideal μ of a lattice is a fuzzy prime ideal if and only if any nonempty α cut set of μ is a prime ideal. In [7], Rajesh kumar studied prime fuzzy ideals over a non - commutative ring. This notion of primeness is equivalent to level cuts being crisp prime ideals. It also generalizes the one provided by Kumbhojkar and Bapat in 1993, which lacks this equivalence in a non-commutative setting. Semiprime fuzzy ideals over a non-commutative ring are also define and characterized as intersection of primes. F.J.Lobillo, O.Cortadellas and G.Navarro [4] have continued the study of semiprime fuzzy ideals over a non-commutative ring and also have reviewed the several definitions of prime fuzzy ideal and semiprime fuzzy ideal.

In this paper, we consider L-Fuzzy Sets where the membership values do not necessarily form a chain. In this general setting we introduce the concept of L-Fuzzy Almost Ideals. So far L-Fuzzy Ideals have been well studied in the case where L is a distributive lattice. We show that the natural extension of concept of L-Fuzzy Ideals to the case when L is not necessarily distributive is that of L- Fuzzy Almost Ideals.

II. PRELIMINARIES

Let X be a nonempty subset, $(L, \leq, \vee, \wedge)$be a complete distributive lattice, which has least and greatest elements, say 0 and 1 respectively. Relevant definitions are recalled in this section.

Definition 2.1 Let X be a nonempty set. A mapping $\mu: X \rightarrow [0,1]$ is called a fuzzy subset of X.

Definition 2.2 Let X be a nonempty set. A mapping $\mu: X \rightarrow L$ is called a L-fuzzy subset of X.

Definition 2.3 A fuzzy subset μ of a ring R is called a fuzzy ideal of R if for all $x,y \in R$,

$\mu(x-y) \geq \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq \mu(x) \vee \mu(y)$.

Definition 2.4 let μ be any fuzzy subset of a set X, $t \in [0,1]$. Then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called a level set of μ. More generally if μ is L-fuzzy set defined by $\mu: X \rightarrow L$ then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called a level set of μ.

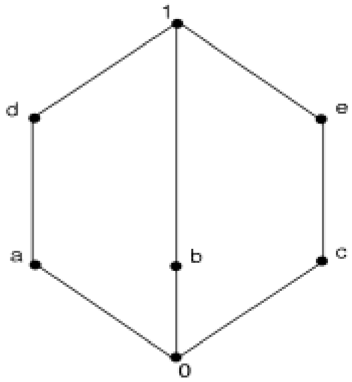
III. L-fuzzy Almost Ideal

Notation Consider $\mu: X \rightarrow L$. If L is totally ordered then for all $x,y \in R$, μ (x) and μ (y) are comparable. That is either $\mu(x) > \mu(y)$ or $\mu(x) = \mu(y)$ or $\mu(x) < \mu(y)$. But if L is not totally ordered then are four possibilities $\mu(x) > \mu(y)$ or $\mu(x) = \mu(y)$ or $\mu(x) < \mu(y)$ or μ (x) and μ (y) are not comparable. We use the notation $\mu(x) \nless \mu(y)$ to mean , " $\mu(x) > \mu(y)$ or $\mu(x) = \mu(y)$ or μ (x) and μ (y) are not comparable".

Definition 3.1 Let R be a ring with unity. Let L be a lattice $(L, \leq, \vee, \wedge)$ not necessarily distributive with least and greatest element 0 and 1 respectively. $\mu: R \rightarrow L$ with $\mu(0) = 1$ and $\mu(1) = 0$ is said to be L-fuzzy almost ideal if

$\mu(x-y) \nless \mu(x) \wedge \mu(y)$ and $\mu(xy) \nless \mu(x)$ and $\mu(xy) \nless \mu(y)$ for all $x,y \in R$.

Example 3.2 The following is an example of a L-Fuzzy Almost Ideals of the ring of integers.



Let R=Z. Let L be a lattice (L, ≤ ,∨,∧) defined by above Hasse diagram. Note that L is not distributive.

Define μ: Z→L as follows.

$$\mu(x) \;=\; \begin{cases} 1 \;\; if\; x = 0 \\ a \;\; if\; x \;\in <p> - <p^2> - <q> - \{0\} \\ d \;\; if\; x \;\in <p^2> - <q> - \{0\} \\ c \;\; if\; x \;\in <q> - <q^2> - <p> - \{0\} \\ e \;\; if\; x \;\in <q^2> - <p> - \{0\} \\ b \;\; if\; x \;\in <pq> - \{0\} \\ 0 \;\; if\; x \;\in Z - <p> - <q> \end{cases}$$

Verification that μ is a L-fuzzy almost ideal :

Let the lattice be L= $C_1 \cup C_2 \cup C_3$ where C_1= {0,a,d,1}, C_2= {0,b,1} and C_3 ={0,c,e,1}. It is required to check the following axioms are satisfied.

Axiom 1 : μ(x-y) ≮ μ(x) ∧ μ(y) for all x,y ∈ R

Axiom 2 : μ(xy) ≮ μ(x) and μ(xy) ≮ μ(y) for all x,y ∈ R.

A systematic investigation of all possibilities reveals that both axioms are indeed satisfied. As an example consider the case μ(x) = μ(y) = a then p | x, p | y therefore p | x - y and p^2 may or may not divide x - y. p^2 | xy. Also q does not divide x, q does not divide y therefore q may or may not divide x - y . q does not divide xy. We have μ(x - y) ∈ {a,d,b} and μ(xy) = d therefore μ(x) ∧ μ(y) = a and μ(x-y) ≮ μ(x) ∧ μ(y) then μ(xy) = d and μ(xy) ≮ μ(x), μ(xy) ≮ μ(y). So in this case both axioms are satisfied. All the cases can be summarized in the form of a table (Table -I).

Remark 3.3 If μ is a L fuzzy ideal, then for any t ∈ Im μ, μ_t is an ideal. However if μ is L-fuzzy almost ideal then in general μ_t need not be an ideal.

Consider the example 3.2. It can be seen that the level set μ_d = {x ∈ Z | μ (x) ≥ d} = $<p^2> - <q>$ is not an ideal.

However for t = 1, we get μ_1 is indeed an ideal.

Theorem 3.4 Let μ: R→L be a L-fuzzy almost ideal then A_1 = {x ∈ R | μ (x) = 1} is an ideal.

Proof :(i) If x , y ∈ A_1 then (x - y) ∈ A_1 :

If x,y ∈ A_1 then μ(x) =1 and μ(y) =1. Since μ(x-y) ≮ μ (x) ∧ μ(y) = 1 ∧ 1, μ (x-y) must be 1. Hence (x-y) ∈ A_1.

(ii) If x ∈ A_1 and y ∈ R, μ(x) = 1.Since μ(xy) ≮ μ (x) = 1, μ(xy) =1.Hence (xy) ∈ A_1.The case x ∈ R, y ∈ A_1 is similar. A_1 is an ideal.

Table - I

μ(x)	μ(y)	μ(x-y)	μ(x) ∧ μ(y)	Axiom 1	μ(xy)	Axiom 2
a	a	a,d or b	A	yes	d	yes
a	d	a or b	a	yes	d	yes
d	d	d or b	d	yes	d	yes
b	b	b	b	yes	b	yes
c	c	c, e or b	c	yes	e	yes
c	e	c or b	c	yes	e	yes
e	e	e or b	e	yes	e	yes
μ(x)	1	μ(x)	μ(x)	yes	1	yes
0	a	0,c,e	0	yes	a	yes
0	d	0,c,e	0	yes	d	yes
0	b	0	0	yes	b	yes
0	c	0,a,d	0	yes	c	yes
0	e	0,a,d	0	yes	e	yes

IV. Almost level set

Definition 4.1 If μ is a L-fuzzy set. We define Almost level set ≈ A_t as, ≈ A_t ={x ∈ R| μ (x)≮ t}.

Proposition 4.2 If t ≤ s, ≈ A_t ⊇ ≈ A_s.

Theorem 4.3 Let R be a ring with unity and (L,≤,∨,∧) a lattice. Let A be a L-fuzzy set defined by μ: R→L. If for all t ∈ Im μ, ≈ A_t is an ideal then A is a L-fuzzy almost ideal.

Proof: Suppose for all t ∈ Im μ, ≈ A_t is an ideal. Let x , y ∈ R. Let μ(x) = t and μ(y) = s. So x ∈ ≈ A_t and y ∈ ≈ A_s.

***Case I*- If t and s are comparable:**

Without loss of generality say t ≤ s. Then by proposition 4.2 ≈ A_t ⊇ ≈ A_s.So y ∈ ≈ A_t.Since x and y belong to ≈ A_t and ≈ A_t is an ideal, x-y ∈ ≈ A_t and xy ∈ ≈ A_t.Therefore μ(x-y) ≮ t.But μ (x) ∧ μ(y) = t ∧ s = t.

So μ(x-y) ≮ μ (x) ∧ μ(y).............(1)

xy ∈ ≈ A_t. So μ(xy) ≮ t = μ (x) ....(2a)

Also y ∈ ≈ A_s and ≈ A_s is an ideal gives xy ∈ ≈ A_s. Therefore μ(xy) ≮ s = μ (y) ....(2b)

***Case II*- If t and s are not comparable:**

Since t and s are not comparable, μ (x) = t means μ(x) ≮ s. So x ∈ ≈ A_s. We already know that x ∈ ≈ A_t . So x ∈ (≈ A_s) ∩ (≈ A_t). Similarly y ∈ (≈ A_s) ∩ (≈ A_t). ≈ A_s ∩ ≈ A_t is an ideal, since it is the intersection of two ideals. Therefore x-y, xy, yx $\in \approx A_s \cap \approx A_t$.If μ(x-y)<μ (x) ∧ μ(y) then μ(x-y)<μ (x)= t which contradicts x-y $\in \approx A_t$. So μ(x-y) ≮ μ (x) ∧ μ(y) ....(1)

xy $\in \approx A_s \cap \approx A_t$ means μ(xy)≮ s and μ(xy)≮t .So μ(xy) ≮ μ(x) and μ(xy) ≮ μ(y) ....(2)

In both cases the two conditions for L-fuzzy almost ideal are satisfied.

Remark 4.4 Converse of the above theorem is not true. In example 3.2 μ is a L-fuzzy almost ideal, but $\approx A_b$ = < p > ∪ < q > is not an ideal.

Theorem 4.5 Let L be the distributive lattice and μ be any L-fuzzy ideal, then μ is an L-fuzzy almost idea. The proof is obvious.

Remark 4.6 Theorem 4.5 shows that the concept of L-fuzzy almost ideal is indeed a generalization of the L-fuzzy ideal.

Remark 4.7 Note that in example 3.2 μ is a L-fuzzy almost ideal but it is not an L-fuzzy ideal.

Theorem 4.8 Let L be a distributive lattice and μ : R → L be any L-fuzzy almost ideal. If Im μ is a chain then μ is an L-fuzzy ideal.

Proof: Let L be a distributive lattice.

Let μ : R → L be any L-fuzzy almost ideal. So

Axiom 1 : μ(x-y) ≮ μ(x) ∧ μ(y) for all x,y ∈ R

Axiom 2*:* μ(xy)≮μ(x) and μ(xy) ≮ μ(y) for all x,y ∈ R.

Since Im μ is a chain it all elements in Im μ are comparable. To prove that μ is a L- fuzzy ideal it is enough to prove that (i) : μ(x-y) ≥ μ(x) ∧ μ(y) and (ii): μ(xy) ≥ μ(x) ∨ μ(y).By axiom (1) μ(x-y) ≮ μ(x) ∧ μ(y).But μ(x-y) and μ(x) ∧ μ(y) are comparable. So μ(x-y) ≥ μ(x) ∧ μ(y).Similarly μ(xy) ≮ μ(x) and μ(xy) ≮ μ(y) gives us the result μ(xy) ≥ μ(x) and μ(xy) ≥ μ(y).Therefore μ(xy)≥μ(x)∨ μ(y).Hence μ is an L-fuzzy ideal.

V CONCLUSION

In this paper the concept of L-fuzzy almost ideal has been introduced and illustrative examples have been given. Some properties have been studied. In our next paper we will extend the concept of primeness to L-fuzzy almost ideals.

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