Observations on the equation $y^2=312x^2+1$

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ABSTRACT

The binary quadratic equation y2=312x2+1 is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few patterns of Pythagorean triangles and rectangles are observed.

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NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

$GF_n(k,s)$: Generalized Fibonacci Sequences of rank n

$GL_n(k,s)$: Generalized Lucas Sequences of rank n

INTRODUCTION:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1,2,3,4].In [5] infinitely many Pythagorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 1$.In [6],a special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [7], different patterns of infinitely many Pythagorean triangles are obtained by employing the non-integral solutions of $y^2 = 12x^2 + 1$.In this context one may also refer [8-14]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 312x^2 + 1$ representing a hyperbola. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration a few patterns of Pythagorean triangles are obtained.

METHODS OF ANALYSIS:

The binary quadratic equation representing a hyperbola is

$$y^2 = 312x^2 + 1 \qquad (1)$$

The smallest positive integer solution to (1) is

$$x_0 = 3, y_0 = 53$$

By applying Brahmagupta method,the general solution of (1) is given by

$$y_n = \frac{1}{2}\left((53+3\sqrt{312})^{n+1} + (53-3\sqrt{312})^{n+1}\right)$$

$$x_n = \frac{1}{2\sqrt{312}}\left((53+3\sqrt{312})^{n+1} - (53-3\sqrt{312})^{n+1}\right)$$

where $n = 0,1,2,3,4..........$

A few numerical examples are presented in the table below:

n	x_n	y_n
0	3	53
1	318	5617
2	33705	595349
3	3272412	63101377
4	378641967	6688150613
5	40132476090	708880863601
6	4253663823573	75134683391093

The recurrence relations satisfied by the values of x_n and y_n are respectively,

$$y_{n+2} - 106y_{n+1} - y_n = 0$$
$$x_{n+2} - 106x_{n+1} - x_n = 0$$

A few interesting relations among the solutions are presented below:

$$2x_{n+2} - 11234x_n \cong 0(\text{mod}312)$$
$$106x_{n+2} - 11234x_{n+1} \cong 0(\text{mod}3)$$

Each of the following expression is a nasty number

- $318[2y_{2n+2} - 1872x_{2n+1} + 106]$
- $6[212y_{2n+2} - 2y_{2n+3} + 2]$
- $18[106x_{2n+3} - 11234x_{2n+2} + 6]$
- $18[2x_{2n+2} - 106x_{2n+1} + 6]$
- $53[2x_{2n+3} - 11234x_{2n+1} + 636]$
- $33702[2y_{2n+3} - 198432x_{2n+1} + 11234]$
- $6[11234y_{2n+3} - 198432x_{2n+3} + 2]$
- $6[106y_{2n+2} - 1872x_{2n+2} + 2]$
- $318[11234y_{2n+2} - 1872x_{2n+3} + 106]$
- $318[106y_{2n+3} - 198432x_{2n+2} + 106]$
- $5298[1872y_{2n+1} + 1872x_{2n+1} - 2y_{2n+2} + 1766]$
- $318[106y_{2n+3} - 198432x_{n+1} + 106]$
- $6[2y_{2n+1} + 2]$

Each of the following expression is a cubical integer:

- $(53)^2[2y_{3n+3} - 11872x_{3n+2} + 318y_n]$
- $[212y_{3n+4} - 2y_{3n+4} + 6y_n]$
- $9[106x_{3n+4} - 11234x_{3n+3} + 18y_n]$
- $9[2x_{3n+3} - 106x_{3n+2} + 18y_n]$
- $(318)^2[2x_{3n+4} - 11234x_{3n+2} + 1908y_n]$
- $(5617)^2[2y_{3n+4} - 198432x_{3n+2} + 33702y_n]$
- $11234y_{3n+4} - 198432x_{3n+4} + 6y_n$
- $106y_{3n+3} - 1872x_{3n+3} + 6y_n$
- $(53)^2[106y_{3n+4} - 198432x_{3n+3} + 318y_n]$
- $(53)^2[11234y_{3n+3} - 1872x_{3n+4} + 318y_n]$
- $(883)^2[1872y_{3n+2} + 1872x_{3n+2} - 2y_{3n+3} + 5299y_n]$
- $(93599)^2[198432y_{3n+2} + 198432x_{3n+2} - 2y_{3n+4} + 561594y_n]$
- $(5299)^2[11234y_{3n+2} + 11234x_{3n+2} - 2x_{3n+4} + 31794y_n]$

Each of the following expression is a biquadratic integer

- $(53)^3[2y_{4n+4} - 1872x_{4n+3} + 848y_n^2 - 106]$
- $212y_{4n+4} - 2y_{4n+5} + 16y_n^2 - 2$
- $3[106x_{4n+5} - 11234x_{4n+4} + 48y_n^2 - 6]$
- $3[2x_{4n+4} - 106x_{4n+3} + 48y_n^2 - 6]$
- $(318)^3[2x_{4n+5} - 11234x_{4n+3} + 5088y_n^2 - 636]$
- $(5617)^3[2y_{4n+5} - 198432x_{4n+3} + 89872y_n^2 - 11234]$
- $11234y_{4n+5} - 198432x_{4n+5} + 16y_n^2 - 2$
- $106y_{4n+4} - 1872x_{4n+4} + 16y_n^2 - 2$
- $(53)^3[11234y_{4n+4} - 1872x_{4n+5} + 848y_n^2 - 106]$
- $(53)^3[106y_{4n+5} - 198432x_{4n+4} + 848y_n^2 - 106]$
- $(883)^3[1872y_{4n+3} + 1872x_{4n+3} - 2y_{4n+4} + 14128y_n^2 - 1766]$
- $(93599)^3[198432y_{4n+3} + 198432x_{4n+3} - 2y_{4n+5} + 1497584y_n^2 - 187198]$
- $(5299)^3[11234y_{4n+3} + 11234x_{4n+3} - 2x_{4n+5} + 84784y_n^2 - 10598]$

Each of the following is a quintic integer:

- $(53)^4[2y_{5n+5} - 1872x_{5n+4} + 2120y_n^3 - 530y_n$
- $212y_{5n+5} - 2y_{5n+6} + 40y_n^3 - 10y_n$
- $(3)^4[106x_{5n+6} - 11234x_{5n+5} + 120y_n^3 - 30y_n]$
- $(3)^4[2x_{5n+5} - 106x_{5n+4} + 120yn^3 - 30y_n]$
- $(318)^4[2x_{5n+6} - 11234x_{5n+4} + 12720y_n^3 - 3180y_n]$
- $(5617)^4[2y_{5n+6} - 198432x_{5n+4} + 224680y_n^3 - 56170y_n]$
- $11234y_{5n+6} - 198432x_{5n+6} + 40y_n^3 - 10y_n]$
- $(53)^4[11234y_{5n+5} - 1872x_{5n+6} + 2120y_n^3 - 530y_n]$
- $(53)^4[11234y_{5n+5} - 1872x_{5n+6} + 2120y_n^3 - 530y_n]$

Remarkable Observation:1

Employing linear combination among the solutions of (1),one may generate integer solution for other choices of hyperbola which are presented in the table below :

HYPERBOLA	(X_n, Y_n)
$11232Y_n^2 - 2X_n^2 = 11232$	$X_n = y_{n+1} - 53y_n$ $Y_n = y_n$
$11232Y_n^2 - 4X_n^2 = 11232$	$X_n = 53y_{n+2} - 5617y_{n+1}$ $Y_n = 212y_{n+1} - 2y_{n+2}$
$12620275Y_n^2 - X_n^2 = 12620275$	$X_n = 2y_{n+2} - 11234y_n$ $Y_n = y_n$
$11236Y_n^2 - 312X_n^2 = 11236$	$X_n = 2x_{n+1} - 6y_n$ $Y_n = y_n$
$126202756Y_n^2 - 312X_n^2 = 126202$	$X_n = 2x_{n+2} - 636y_n$ $Y_n = y_n$
$4Y_n^2 - 312X_n^2 = 4$	$X_n = 106x_{n+1} - 6y_{n+1}$ $Y_n = 53y_{n+1} - 936x_{n+1}$
$4Y_n^2 - 2808X_n^2 = 36$	$X_n = 212x_{n+1} - 2y_{n+2}$ $Y_n = 53x_{n+2} - 5617x_{n+1}$
$11236Y_n^2 - 312X_n^2 = 11236$	$X_n = 6y_{n+2} - 11234x_{n+1}$ $Y_n = 3744x_{n+1} - 2y_{n+2}$
$4Y_n^2 - 312X_n^2 = 4$	$X_n = 11234x_{n+2} - 636y_{n+2}$ $Y_n = 5617y_{n+2} - 198432x_{n+2}$
$4Y_n^2 - 11232X_n^2 = 36$	$X_n = x_n$ $Y_n = x_{n+1} - 53x_n$
$4Y_n^2 = 126202752X_n^2 = 404496$	$X_n = x_n$ $Y_n = x_{n+2} - 5617x_n$
$Y_n^2 = 876408X_n^2 = 11236$	$X_n = 12x_{n+1} - 2x_{n+2}$ $Y_n = 1872x_{n+2} - 11234y_{n+1}$

Remarkable Observation:2

Employing linear combination among the solutions of (1),one may generate integer solution for other choices of parabola which are presented in the table below:

PARABOLA	(X,Y)
$5616Y_n - 4X_n^2 = 11232$	$Y_n = y_{2n+1} + 1$ $X_n = y_{n+1} - 53y_n$
$5616Y_n - 4X_n^2 = 11232$	$Y_n = y_{2n+1} + 1$ $X_n = 53y_{n+2} - 5617y_n$
$5618Y_n - 312X_n^2 = 11236$	$Y_n = y_{2n+1} + 1$ $X_n = 53y_{n+2} - 5617y_{n+1}$
$2Y_n - 66144X_n^2 = 212$	$Y_n = y_{2n+2} - 936x_{2n+2} + 53$ $X_n = x$
$2Y_n - 1248X_n^2 = 4$	$Y_n = 53y_{2n+2} - 936x_{2n+2}$ $X_n = 106x_{n+1} - 6y_{n+1}$
$31550689Y_n - 312X_n^2 = 126202756$	$Y_n = y_{2n+1} + 1$ $X_n = 212x_{n+1} - 2x_{n+2}$
$5618Y_n - 312X_n^2 = 11236$	$Y_n = y_{2n+3} - 1872x_{2n+2}$ $X_n = x_n$
$2Y_n - 3744X_n^2 = 12$	$Y_n = x_{2n+2} - 53x_{2n+1} + 3$ $X_n = 11234x_{n+1} - 6y_{n+2}$
$2Y_n - 396864X_n^2 = 1272$	$Y_n = x_{2n+3} - 5617x_{2n+1} + 318$ $X_n = x_n$
$2Y_n - 936X_n^2 = 12$	$Y_n = 53x_{2n+3} - 5617x_{2n+2} + 3$ $X_n = x_n$
$5616Y_n - 4x_n^2 = 11232$	$Y_n = y_{2n+1} + 1$ $X_n = x_n$
$Y_n - 496X_n^2 = 8$	$Y_n = 63x_{2n+3} - 7937x_{2n+2} + 4$ $X_n = 25x_{n+1} - 2x_{n+2}$

Remarkable Observation:3

Employing linear combination among the solutions of (1),one may generate integer solution for other choices of straight line which are presented in the table below:

STRAIGHT LINE	(X,Y)
$Y = 53X$	$X = 2y_n$ $Y = 2y_{n+1} - 1872x_n$
$Y = 53X$	$X = 212y_{n+1} - 2y_{n+2}$ $Y = 2y_{n+1} - 1872x_n$
$Y = 3X$	$X = 212y_{n+1} - 2y_{n+2}$ $Y = 106x_{n+2} - 11234x_{n+1}$
$Y = 3X$	$X = 312y_{n+1} - 2y_{n+2}$ $Y = 2x_{n+1} - 106x_n$
$Y = 318X$	$X = 212y_{n+1} - 2y_{n+2}$ $Y = 2x_{n+2} - 11234x_n$

$Y = 53X$	$X = 212y_{n+1} - 2y_{n+2}$ $Y = 11234y_{n+1} - 1872x_{n+2}$
$Y = 883X$	$X = 212y_{n+1} - 2y_{n+2}$ $Y = 1872y_n + 1872x_n - 2y_{n+1}$
$Y = 53X$	$X = 212y_{n+1} - 2y_{n+2}$ $Y = 106y_{n+2} - 198432x_{n+1}$

Remarkable Observation:4

(i) Let (x, y) be any given non-zero positive integer solutions of (1).Let r and s any non-zero distinct positive integers. Choose r and s such that $r > s$ and $r - s = y$. Taking r and s to be the generators of the Pythagorean triangle (α, β, γ) .where $\alpha = 2rs, \beta = r^2 - s^2, \gamma = r^2 + s^2$ observe the relations $155\gamma + \alpha - 156\beta + 1 = 0$

(ii) Let m, n be any non-zero distinct positive integers defined by

$m = x, n = \frac{y-1}{2}$, then it is observed that

$4t_{3,n}(t_{3,m})^2 = 156(P_m^5)^2$

(iii) The solutions of (1) interms of special integers namely,Generalized Fibonacci GF_n and Lucas GL_n numbers are exhibited below:

$x_n = 3GF_{n+1}(106, -1)$

$y_n = \frac{GL_{n+1}}{2}(106, -1)$

CONCLUSION:

To conclude, one may search for other choices of positive Pell equations and negative pell equations for finding their integer solutions.

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