Differential Subordination with Hadamard Product of Generalized k-Mittag-Leffler Function and a Class of Function

[1]Prakash Chand Goyal, [2]Ashok Singh Shekhawat
[1]*Department of Mathematics, Suresh Gyan Vihar University, Jagatpura, Jaipur-302017, India.*
[2]*Department of Mathematics, Suresh Gyan Vihar University, Jagatpura, Jaipur-302017, India.*

Abstract
In this paper we introduce Differential subordination with Hadamard Product (Convolution) of Generalized k-Mittag-Leffler function and A Class of Function in the Open Unit Disk $\mathbb{D}= \{z \in \mathbb{C} : |z| < 1\}$, *Which are expressed in terms of the A Class of Function. Some interesting special cases of our main results are also considered.*

AMS subject classification: 30C45, 30C80, 33E12.
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I. INTRODUCTION

In 1903, the Swedish mathematician Gosta Mittag-Leffler [13] (see also [14]) introduced and investigated the so-called Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad \dots (1.1)$$

Where $z \in \mathbb{C}, \alpha \in \mathbb{C}; Re(\alpha) > 0,\ \Gamma$ represents well known Gamma function.

Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [2], [5], [6], [7], [8], [10], [15], [16], [17], [18], [26], [27] and [28].

The generalization of $E_\alpha(z)$, also known as Wiman function, is given by Wiman [28]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad \dots (1.2)$$

where $\alpha, \beta \in \mathbb{C},; Re(\alpha) > 0$and $Re(\beta) > 0$.

Further, in 1971, Prabhakar [17] proposed the more general function $E^{\gamma}_{\alpha,\beta}(z)$ as:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\ \Gamma(\alpha n + \beta)} z^n \qquad \dots (1.3)$$

for which $\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0$ and $Re(\gamma) > 0$. The importance and great considerations of Mittag-Leffler function have led many researchers in the theory of special functions for exploring the possible generalizations and applications. Many more extensions or unifications for these functions are found in large number of papers [4], [20], [21], [23] and [25]. A useful generalization of the Mittag-Leffler function called as k-Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$, introduced in [6], and it is given by

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{n!\ \Gamma_k(\alpha n + \beta)} z^n \qquad \dots (1.4)$$

where $\alpha, \beta, \gamma \in \mathbb{C},\ k \in \mathbb{R}, \{Re(\alpha) > 0, Re(\beta) > 0$ and $Re(\gamma) > 0\}$ and $(\gamma)_{n,k}$ is the k-Pochhammer symbol defined as:

$$(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k) \dots (\gamma + (n-1)k) \qquad \dots (1.5)$$

Where $\gamma \in \mathbb{C},\ k \in \mathbb{R}, n \in \mathbb{N}$.

Lately, a generalized form of k-Mittag-Leffler function was introduced and studied in [3] as:
Let $\alpha,\beta,\gamma \in \mathbb{C},\ k \in \mathbb{R}, \{Re(\alpha) > 0, Re(\beta) > 0$ and $Re(\gamma) > 0\}$ and $q \in (0,1) \cup \mathbb{N}$, then

$$E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \qquad \text{... (1.6)}$$

where $(\gamma)_{nq,k}$ is defined as (1.5) and the generalized Pochhammer symbol is defined as (see [19]):

$$(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)} \qquad \text{... (1.7)}$$

In the integral representation, the generalized k-Gamma function is defined as:

$$\Gamma_k(n) = \int_0^\infty e^{\frac{-t^k}{k}} t^{n-1} dt \qquad \text{... (1.8)}$$

Let A denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{... (1.9)}$$

which are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Srivastava and Tomovski [25] proved that the generalized Mittag-Leffler function is an entire function in the complex z-plane. Using Srivastava and Tomovski [25, theorem 1, P-201], we find that, if $Re(\alpha) \geq 0$ when $Re(q) \geq 0$ with $\beta \neq 0$, Then, the power series in the defining equation (1.6) is still analytic and converges absolutely in open unit disc $\mathbb{D}$ for all $\gamma \in \mathbb{C}$.

Shekhawat and Goyal [22] defined that Hadamard product (Convolution) of Generalized k-Mittag-Leffler function $\Im_{k,\alpha,\beta}^{\gamma,q}(z)$ and A Class of Function $f(z)$ is

$$\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)} \frac{\Gamma\left(\frac{\gamma}{k}\right)\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q-\left(\frac{\alpha+\beta}{k}\right)}\Gamma\left(\frac{\gamma+kq}{k}\right)} \frac{a_n}{n!} z^n \qquad \text{... (1.10)}$$

where

- $\alpha,\beta,\gamma \in \mathbb{C},\ k \in \mathbb{R}, \{Re(\alpha) > 0, Re(\beta) > 0$ and $Re(\gamma) > 0\}$ and $q \in (0,1) \cup \mathbb{N}$.
- $(\gamma)_{nq,k}$ is generalized Pochhammer symbol.
- $f(z)$ is analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
- $\beta,\gamma \in \mathbb{C};\ Re(\alpha) > max\{0, Re(q) - 1\}, Re(q) > 0$ and $Re(\alpha) = 0$ when $Re(q) = 1$ with $\beta,\gamma \neq 0$.

Let $\psi: \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ be analytic in domain $\mathbb{D}$, and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ with $[p(z), zp'(z)] \in \mathbb{D}$ when $z \in \mathbb{U}$, then we say that $p(z)$ satisfies a first order differential subordination if:

$$\psi[p(z), zp^{'}(z); z] \prec h(z) \qquad \text{... (1.11)}$$

The univalent function $q(z)$ is called dominant of the differential subordination (1.11); if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.11), if $\bar{q}(z) \prec q(z)$ for all dominant of (1.11), then we say that $\bar{q}(z)$ is the best dominant of (1.11).

Lemma-1 Let $q(z)$ be univalent in $\mathbb{U}$ and let θ and $\emptyset$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$, with $\emptyset(w) \neq 0$, when $w \in q(\mathbb{U})$. Set $Q(z) = zq^{'}(z).\emptyset[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either $h(z)$ is convex, or $Q(z)$ is strictly In addition, assume that

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0.$$

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp^{'}(z).\emptyset[p(z)]q(z) \prec \theta[q(z)] + zq^{'}(z).\emptyset[q(z)] = h(z), \qquad (1.12)$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.12).

Proof: this lemma proof by Miller and Mocanu [11],[12].

Lemma-2 If $f(z) \in A$ is analytic in the open unit disc $\mathbb{D}= \{z \in \mathbb{C} : |z| < 1\}$ then

$$z\left(\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z)\right)' = \left(\frac{\gamma + kq}{kq}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma+1,q}(f)(z)\right) - \frac{\gamma}{kq}\left(\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z)\right) \qquad ...(1.13)$$

Proof: By using the definition of $\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z)$ which is defined by (1.10)

$$\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n + \beta)} \frac{\Gamma\left(\frac{\gamma}{k}\right)\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q-\left(\frac{\alpha+\beta}{k}\right)}\,\Gamma\left(\frac{\gamma+kq}{k}\right)} \frac{a_n}{n!} z^n$$

Using Pochhammer symbol

$$(\gamma)_{nq,k} = \frac{k^{nq}.\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma\left(\frac{\gamma}{k}\right)}$$

then

$$\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma_k(\alpha n + \beta)} \frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\,\Gamma\left(\frac{\gamma+kq}{k}\right)} \frac{a_n}{n!} z^n$$

Differentiating

$$\left(\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z)\right)' = 1 + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma_k(\alpha n + \beta)} \frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\,\Gamma\left(\frac{\gamma+kq}{k}\right)} \frac{a_n}{n!} nz^{n-1}$$

Multiplying by z

$$z\left(\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z)\right)' = z + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma_k(\alpha n + \beta)} \frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\,\Gamma\left(\frac{\gamma+kq}{k}\right)} \frac{a_n}{n!} nz^n$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma_k(\alpha n + \beta)} \frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\,\Gamma\left(\frac{\gamma+kq}{k}\right)} \frac{a_n}{n!} z^n \left[\frac{\gamma+kqn}{kq} - \frac{\gamma}{kq}\right]$$

$$= z + \frac{1}{q}\left(\sum_{n=2}^{\infty}\frac{\left(\frac{\gamma+kqn}{k}\right).\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ \Gamma\left(\frac{\gamma+kq}{k}\right)}\frac{a_n}{n!}z^n\right) - \frac{\gamma}{kq}\sum_{n=2}^{\infty}\frac{\Gamma\left(\frac{\gamma+knq}{k}\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ \Gamma\left(\frac{\gamma+kq}{k}\right)}\frac{a_n}{n!}z^n$$

$$= z\left(\frac{\gamma+kq}{kq}-\frac{\gamma}{kq}\right) + \left(\frac{\gamma+kq}{kq}\right)\left(\sum_{n=2}^{\infty}\frac{\left(\frac{\gamma+kqn}{k}\right).\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ .\left(\frac{\gamma+kq}{k}\right).\Gamma\left(\frac{\gamma+kq}{k}\right)}\frac{a_n}{n!}z^n\right) - \frac{\gamma}{kq}\sum_{n=2}^{\infty}\frac{\Gamma\left(\frac{\gamma+knq}{k}\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ \Gamma\left(\frac{\gamma+kq}{k}\right)}\frac{a_n}{n!}z^n$$

$$= \frac{\gamma+kq}{kq}\left(z+\sum_{n=2}^{\infty}\frac{\left(\frac{\gamma+kqn}{k}\right).\Gamma\left(\frac{\gamma+kqn}{k}\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ .\left(\frac{\gamma+kq}{k}\right).\Gamma\left(\frac{\gamma+kq}{k}\right)}\frac{a_n}{n!}z^n\right) - \frac{\gamma}{kq}\left(z+\sum_{n=2}^{\infty}\frac{\Gamma\left(\frac{\gamma+knq}{k}\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ \Gamma\left(\frac{\gamma+kq}{k}\right)}\frac{a_n}{n!}z^n\right)$$

$$= \frac{\gamma+kq}{kq}\left(z+\sum_{n=2}^{\infty}\frac{\Gamma\left(\frac{\gamma+kqn}{k}+1\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ .\Gamma\left(\frac{\gamma+kq}{k}+1\right)}\frac{a_n}{n!}z^n\right) - \frac{\gamma}{kq}\left(z+\sum_{n=2}^{\infty}\frac{\Gamma\left(\frac{\gamma+knq}{k}\right)}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)}{k^{1+q(1-n)-\left(\frac{\alpha+\beta}{k}\right)}\ \Gamma\left(\frac{\gamma+kq}{k}\right)}\frac{a_n}{n!}z^n\right)$$

$$= \left(\frac{\gamma+kq}{kq}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma+1,q}(f)(z)\right) - \frac{\gamma}{kq}\left(\aleph_{k,\alpha,\beta}^{\gamma,q}(f)(z)\right).$$

Which is the required result.

II. MAIN RESULT

Theorem: Let

$$\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)} < q(z) + \frac{z\big(q(z)\big)'}{q(z)+\frac{\gamma}{k\eta}}$$

And

$$\frac{\aleph_{k,\alpha,\beta}^{\gamma+i,\eta}(f)(z)}{z} \neq 0\ (i = 0,1$$

$f(z) \in A$ is analytic in the open unit disc $\mathbb{D}= \{z \in \mathbb{C} :\ |z| < 1\}$ and $q(z)$ is univalent in $\mathbb{D}$ with $q(0) = 1$, which satisfies the following conditions:

$$Re\left(q(z)+\frac{\gamma}{k\eta}\right)>0$$

And

$$Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)+\frac{\gamma}{k\eta}}\right)>0$$

Then

$$\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)}\prec q(z) \qquad ...(2.1)$$

And $q(z)$ is the best dominant of (2.1).

Provided that:
$\alpha,\beta,\gamma\in\mathbb{C},\ k\in\mathbb{R},\{Re(\alpha)>0, Re(\beta)>0$ and $Re(\gamma)>0\}$ and $\eta\in(0,1)\cup\mathbb{N}$.

Proof: from lemma-2

$$z\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'=\left(\frac{\gamma+k\eta}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)-\frac{\gamma}{k\eta}\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)$$

Dividing by $\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)$, we have

$$\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)}=\left(\frac{\gamma+k\eta}{k\eta}\right)\frac{\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)}{\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)}-\frac{\gamma}{k\eta}$$

Let

$$p(z)=\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)}$$

Then

$$p(z)+\frac{\gamma}{k\eta}=\left(\frac{\gamma+k\eta}{k\eta}\right)\frac{\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)}{\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)}$$

$$\Rightarrow\left(p(z)+\frac{\gamma}{k\eta}\right)\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)=\left(\frac{\gamma+k\eta}{k\eta}\right)\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z) \qquad ...(2.2)$$

Taking differentiation

$$\left(p(z)+\frac{\gamma}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'+p'(z)\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)=\left(\frac{\gamma+k\eta}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'$$

or $$z\left(p(z)+\frac{\gamma}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'+zp'(z)\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)=z\left(\frac{\gamma+k\eta}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'$$

or $$z\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'+\frac{zp'(z)}{\left(p(z)+\frac{\gamma}{k\eta}\right)}\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)=\frac{z\left(\frac{\gamma+k\eta}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'}{\left(p(z)+\frac{\gamma}{k\eta}\right)}$$

or $$\frac{z\left(\frac{\gamma+k\eta}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'}{\left(p(z)+\frac{\gamma}{k\eta}\right)\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)}=p(z)+\frac{zp'(z)}{\left(p(z)+\frac{\gamma}{k\eta}\right)}$$

Using (2.2)

$$\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)} = p(z) + \frac{zp'(z)}{\left(p(z) + \frac{\gamma}{k\eta}\right)} \qquad \ldots (2.3)$$

Given that

$$\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)} \prec q(z) + \frac{z\left(q(z)\right)'}{q(z) + \frac{\gamma}{k\eta}} \qquad \ldots (2.4)$$

Where $q(z)$ is defined in theorem.

From (2.3) and (2.4) we have

$$p(z) + \frac{zp'(z)}{\left(p(z) + \frac{\gamma}{k\eta}\right)} \prec q(z) + \frac{z\left(q(z)\right)'}{q(z) + \frac{\gamma}{k\eta}}$$

By Miller and Mocanu [11] lemma
We deduce

$$\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)} \prec q(z) \qquad \ldots (2.5)$$

And $q(z)$ is the best dominant of (2.5).

Corollary: Let

$$Re\left(\frac{\gamma}{k\eta}\right) \geq -\delta;\ \delta \in [0,1).$$

Also, let

$$\frac{z\left(\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)\right)'}{\aleph_{k,\alpha,\beta}^{\gamma+1,\eta}(f)(z)} \prec h(z),\ \ \forall f \in A.$$

satisfies

$$\frac{\aleph_{k,\alpha,\beta}^{\gamma+i,\eta}(f)(z)}{z} \neq 0\ (i = 0,1)$$

then $\left[\aleph_{k,\alpha,\beta}^{\gamma,\eta}(f)(z)\right] \in S^*(\delta)$, δ is the best possible, where $S^*(\delta)$ is starlike function of order δ and

$$h(z) = -1 + 2\delta - \frac{3 - 2\delta}{1 - z} - \frac{1 + \frac{\gamma}{k\eta}}{1 + \frac{\gamma}{k\eta} + \left(1 - 2\delta - \frac{\gamma}{k\eta}\right)z}.$$

Proof: Putting

$$q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z},$$

therefore under the condition

$$Re\left(\frac{\gamma}{k\eta}\right) \geq -\delta;\ \delta \in [0,1),$$

We have

$$Re\left(q(z) + \frac{\gamma}{k\eta}\right) > 0.$$

After some calculation, we have

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)+\frac{\gamma}{k\eta}}$$

$$=-1+\frac{1}{1-z}+\frac{1+\frac{\gamma}{k\eta}}{1+\frac{\gamma}{k\eta}+\left(1-2\delta-\frac{\gamma}{k\eta}\right)z}$$

Therefore

$$Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)+\frac{\gamma}{k\eta}}\right)>0.$$

III. CONCLUSION

In this paper we have presented differential subordination Hadamard Product (Convolution) of Generalized k-Mittag-Leffler function and A Class of Function in the Open Unit Disk $\mathbb{D}=\{z\in\mathbb{C}: |z|<1\}$. The result so established may be found useful in several interesting situation appearing in the literature on mathematical analysis. Further many known and unknown results have been established in terms of the A Class of Function. The results presented in this paper are easily converted in terms of the generalized Mittag-leffler function. We are also trying to find certain possible applications of those results presented here to some other research areas.

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REFERENCES

[1] Dorrego, G. A., and Cerutti, R. A., 2012, “The k-Mittag-Leffler function,” Int. J. Contemp. Math. Sci., 7, pp. 705-716.
[2] Garg, M., Manoha, P., and Kalla, S.L., 2013, “A Mittag-Leffrer-type function of two variables,” Integral Transforms Spec. Funct., 24(11), pp. 934–944.
[3] Gehlot, K. S., 2012, “The Generalized k-Mittag-Leffler function,” Int. J. Contemp. Math. Sci., 7, pp. 2213-2219.
[4] Khan, M. A., and Ahmed, S., 2013, “On some properties of the generalized Mittag-Leffler function,” Springer Plus, 2, pp. 1-9.
[5] Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J., 2006, “Theory and Applications of Fractional Differential Equations,” North-Holland Mathematics Studies, 204.
[6] Kiryakova, V., 1994, “Generalized fractional calculus and applications,” Pitman Research Notes in Mathematics Series, 301. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York.
[7] Kiryakova, V., 2000, “Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus, Higher transcendental functions and their applications,” J. Comput. Appl. Math., 118, pp. 1-2, 241–259.
[8] Kiryakova, V., 2010, “The multi-index Mittag-Le_er functions as an important class of special functions of fractional calculus,” Comput. Math. Appl. 59 (5), pp. 1885–1895.
[9] Kiryakova, V., 2011, “Criteria for univalence of the Dziok-Srivastava and the Srivastava-Wright operators in the class A,” Appl. Math. Comput. 218 (3), pp. 883–892.
[10] Mainardia, F., and Gorenflo, R., 2000, “On Mittag-Leffler-type functions in fractional evolution processes.” J. Comput. Appl. Math., 118, pp. 283–299.
[11] Miller, S.S., and Mocanu, P.T., 1985, “On some classes of first-order differential subordinations,” Michigan Math. J. 32, no.2, pp.185-195.
[12] Miller, S.S., and Mocanu, P.T., 2000, “Differential Subordinations: Theory and Applications, Series in Pure and Applied Mathematics,” No.225. Marcel Dekker, Inc., New York.
[13] Mittag-Leffler, G. M., 1903, “Sur la nouvelle function $E_\alpha(x)$,” CR Acad. Sci. Paris, 137, pp. 554-558.
[14] Mittag-Leffler, G. M., 1905, “Sur la representation analytique d'une branche uniforme d'une function monogene,” Acta Math., 29, pp. 101-181.
[15] Ozarslan, M. A., and Yılmaz, B., 2014, “The extended Mittag-Leffler function and its properties,” J. Inequal. Appl., 10 pp.
[16] Podlubny, I., 1999, Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA. 198.
[17] Prabhakar, T. R., 1971, “A singular integral equation with a generalized Mittag-Leffler function in the Kernel,” Yokohama Math. J., 19, pp. 7-15.
[18] Prajapati, J.C., Jana, R.K., Saxena, R.K., and Shukla, A.K., 2013, “Some results on the generalized Mittag-Leffler function operator,” J. Inequal. Appl. 2013(6) pp
[19] Rainville, E. D., 1960, Special Functions, The Macmillan Co., New York.
[20] Salim, T. O., 2009, “Some properties relating to the generalized Mittag-Leffler function,” Adv. Appl. Math. Anal., 4, pp. 21-30.
[21] Salim, T. O. and Ahmad, W. F., 2012, “A generalization of integral operator associated with fractional calculus,” J. Fract. Calc. Appl., 3, pp. 1-13.
[22] Shekhawat, A.S., Goyal, P.C., 2017, “Hadamard Product (Convolution) of Generalized k-Mittag-Leffler Function and A Class of Function” Global J. of Pure and appl. Math. 13(5), pp. 1473-1483.
[23] Shukla, A. K., and Prajapati, J. C., 2007, “On a generalization of Mittag-Leffler function and its properties,” J. Math. Anal. Appl., 336, pp. 797-811.

[24] Srivastava, H.M., 2007, "Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators," Appl. Anal. Discrete Math. 1(1), pp. 56-71.

[25] Srivastava, H. M., and Tomovski, Z., 2009, "Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel," Appl. Math. Comput., 211, pp. 198-210.

[26] Tomovski, Z., Hilfer, R., and Srivastava, H.M., Srivastava, 2010, "Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions," Integral Transforms Spec. Funct. 21 (11), pp. 797–814.

[27] Tomovski, Z., 2012, "Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator," Nonlinear Anal. 75 (7), pp. 3364–3384.

[28] Wiman, A., 1905, "Uber de fundamental satz in der theorie der funktionen $E_a(x)$," Acta Math., 29, pp. 191-201.

1.PRAKASH CHAND GOYAL
Department of Mathematics, Suresh Gyan Vihar University, Jagatpura, Jaipur-302025
Postal Address: I-91, J.P. Colony, Sector No. 3, Shastri nagar, Jaipur, Rajasthan, India-302016

2. ASHOK SINGH SHEKHAWAT
Department of Mathematics, Suresh Gyan Vihar University, Jagatpura, Jaipur-302025