POSNER'S FIRST THEOREM FOR IDEALS IN PRIME RINGS

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Abstract: Posner's first theorem states that if R is a prime ring of characteristic different from two, d_1 and d_2 are derivations on R such that the iterate d_1d_2 is also a derivation of R, then at least one of d_1, d_2 is zero. In the present paper we extend this result for ideals in prime rings of characteristic different from 2.

1. INTRODUCTION

Throughout the paper, R will represent an associative ring. R is called a prime ring if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semiprime if $xRx = \{0\}$ implies $x = 0$. Given an integer $n > 1$, ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let I be a nonzero ideal of R. Then an additive mapping $d : I \longrightarrow R$ is called a derivation from I to R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in I$. In the year 1957, E. C. Posner initiated the study of derivations in rings and proved two very important theorems. These results have been generalized by several authors in different directions see [2,3,4] for reference where further references can be found. Posner's first theorem [5, Theorem 1] states that if R is a prime ring of characteristic not 2 and iterate of two derivations is also a derivation, then at least one of them is zero. In this paper we extend this result for ideals in prime rings of characteristic different from 2.

2. PRELIMINARY RESULT

We begin with the following lemma which is essential for developing the proof of our main result.

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Lemma 2.1. Let I be a nonzero ideal of prime ring R and $d : I \longrightarrow R$ be a derivation. If a is an element of R and $ad(x) = 0$ (resp. $d(x)a = 0$) for all $x \in I$, then either $a = 0$ or $d = 0$.

Proof. Replacing x by xy, where $y \in I$ in the relation $ad(x) = 0$, we obtain that $ad(x)y + axd(y) = 0$ i.e.; $axd(y) = 0$ for all $x, y \in I$. Now replacing x by xt, where $t \in R$ in the last relation we obtain that $axtd(y) = 0$ i.e.; $axRd(y) = \{0\}$ for all $x, y \in I$. Now primeness of R forces either $ax = 0$ for all $x \in I$ or $d = 0$. Suppose that $ax = 0$ for all $x \in I$. Since $I \neq \{0\}$, primeness of R again forces that $a = 0$. Finally, we conclude that either $a = 0$ or $d = 0$. Similarly we can also show that $d(x)a = 0$ for all $x \in I$ implies that either $a = 0$ or $d = 0$.

3. MAIN RESULT

Theorem 3.1. Let R be a prime ring of characteristic not 2, I a nonzero ideal of R and $d_1, d_2 : I \longrightarrow R$ derivations such that the product map $d_1 d_2 : I \longrightarrow R$ is also a derivation, then at least one of d_1 and d_2 is zero.

Proof. Since the map $d_1 d_2 : I \longrightarrow R$ is a derivation, it is obvious that $d_2(I) \subseteq I$ and $d_1 d_2(xy) = d_1 d_2(x)y + x d_1 d_2(y)$ for all $x, y \in I$. As $d_1, d_2 : I \longrightarrow R$ are derivations, we also obtain that

$$\begin{aligned} d_1 d_2(xy) &= d_1(d_2(xy)) \\ &= d_1(d_2(x)y + x d_2(y)) \\ &= d_1 d_2(x)y + d_2(x) d_1(y) + d_1(x) d_2(y) + x d_1 d_2(y). \end{aligned}$$

so by above relations we conclude that

$$d_2(x) d_1(y) + d_1(x) d_2(y) = 0 \tag{3.1}$$

for all $x, y \in I$. Now replacing y by $d_2(y)z$, where $z \in I$ in the relation (3.1) we obtain that $d_2(x) d_1(d_2(y)z) + d_1(x) d_2(d_2(y)z) = 0$ for all $x, y, z \in I$. This gives us $d_2(x) d_1 d_2(y) z + d_2(x) d_2(y) d_1(z) + d_1(x) d_2^2(y) z + d_1(x) d_2(y) d_2(z) = 0$. Now $(d_2(x) d_1 d_2(y) + d_1(x) d_2(d_2(y)))z = 0$, since $(d_2(x) d_1 d_2(y) + d_1(x) d_2(d_2(y)))$ which is merely equation (3.1) with y replaced by $d_2(y)$ and using the fact that $d_2(I) \subseteq I$. Then we are left with

$$d_2(x) d_2(y) d_1(z) + d_1(x) d_2(y) d_2(z) = 0 \tag{3.2}$$

for all $x, y, z \in I$. Now using the relation (3.1) and the fact that R has characteristic different from 2, the relation (3.2) takes the form $d_1(x) d_1(y) d_2(z) = 0$ for all $x, y, z \in I$. Now Lemma 2.1 provides us either $d_2 = 0$ or $d_1(x) d_1(y) = 0$ for all $x, y \in I$. If first case holds then nothing to do, if not we have $d_1(x)d_1(y) = 0$ for all $x, y \in I$. Using Lemma 2.1 again we conclude that $d_1 = 0$.

Now taking $I = R$ in the above theorem we obtain the following:

Corollary 3.1 ([5], Theorem 1). Let R be a prime ring of characteristic not 2 and d_1, d_2 derivations of R such that the iterate d_1d_2 is also a derivation, then at least one of d_1, d_2 is zero.

The following example shows that the hypothesis of primeness is crucial in the above theorem.

Example 3.1. Let $R = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mid x, y, z, 0 \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of integers. It is easy to verify that characteristic of R is different from 2. Further if we set $I = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid y, 0 \in \mathbb{Z} \right\}$, then I is a nonzero ideal of R. Now consider the maps $d_1, d_2, : I \longrightarrow R$ defined by

$$d_1 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, d_2 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -y & 0 \end{pmatrix}.$$

Then it is obvious to observe that d_1 and d_2 are derivations of R. Further it can be also shown that the map $d_1d_2 : I \longrightarrow R$ is a derivation and R is not a prime ring. However neither $d_1 = 0$ nor $d_2 = 0$.

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