Generalized Multiplicative Derivations in Near-Rings

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Abstract: In the present paper, we investigate the commutativity of 3-prime near-rings satisfying certain conditions and identities involving left generalized multiplicative derivations. Moreover, examples have been provided to justify the necessity of 3-primeness condition in the hypotheses of various results.

1. Introduction

Throughout the paper, $\mathcal{N}$ will denote a left near-ring. $\mathcal{N}$ is called a 3-prime near-ring if $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. $\mathcal{N}$ is called a semiprime near-ring if $x\mathcal{N}x = \{0\}$ implies $x = 0$. A nonempty subset $\mathfrak{A}$ of $\mathcal{N}$ is called a semigroup left ideal (resp. semigroup right ideal) if $\mathcal{N}\mathfrak{A} \subseteq \mathfrak{A}$ (resp. $\mathfrak{A}\mathcal{N} \subseteq \mathfrak{A}$) and if $\mathfrak{A}$ is both a semigroup left ideal as well as a semigroup right ideal, it will be called a semigroup ideal of $\mathcal{N}$. The symbol Z will denote the multiplicative center of $\mathcal{N}$, that is, $Z = \{x \in \mathcal{N} \mid xy = yx$ for all $y \in \mathcal{N}\}$. For any $x, y \in \mathcal{N}$ the symbol $[x, y] = xy - yx$ stands for the multiplicative commutator of x and y, while the symbol xoy stands for $xy + yx$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called a derivation of $\mathcal{N}$ if $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in \mathcal{N}$. The concept of derivation has been generalized in different directions by various authors (for reference see $[1, 3, 9]$). A map $d : \mathcal{N} \to \mathcal{N}$ is called a multiplicative derivation of $\mathcal{N}$ if $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in \mathcal{N}$. We, together with M. Ashraf and A. Boua have generalized the notion of multiplicative derivation by introducing the notion of generalized multiplicative derivations in [1] as follows: A map $f : \mathcal{N} \longrightarrow \mathcal{N}$ is called a left generalized multiplicative derivation of $\mathcal{N}$ if there exists a multiplicative derivation d of $\mathcal{N}$ such that $f(xy) = xf(y) + d(x)y$ for all $x, y \in \mathcal{N}$. The map f will be called a left generalized multiplicative derivation of $\mathcal{N}$ with associated multiplicative derivation d of $\mathcal{N}$. Similarly a map $f : \mathcal{N} \longrightarrow \mathcal{N}$ is called a right generalized multiplicative derivation of $\mathcal{N}$ if there exists a multiplicative derivation d of $\mathcal{N}$ such that $f(xy) = xd(y) + f(x)y$ for all $x, y \in \mathcal{N}$. The map f will be called a right generalized multiplicative derivation of $\mathcal{N}$ with associated multiplicative derivation d of $\mathcal{N}$. Finally, a map $f : \mathcal{N} \longrightarrow \mathcal{N}$ will be called a

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generalized multiplicative derivation of $\mathcal{N}$ if it is both a right as well as a left generalized multiplicative derivation of $\mathcal{N}$ with associated multiplicative derivation d of $\mathcal{N}$. Note that if in the above definition both d and f are assumed to be additive mappings, then f is said to be a generalized derivation with associated derivation d of $\mathcal{N}$. The following example shows that there exists a left generalized multiplicative derivation which is not a right generalized multiplicative derivation. For more properties of generalized multiplicative derivations one can refer to [1].

Example 1.1. Let S be a zero-symmetric left near-ring. Suppose that

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}.$$

It can be easily shown that $\mathcal{N}$ is a zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $d, f : \mathcal{N} \longrightarrow \mathcal{N}$ such that

$$d \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x^2 & 0 & 0 \end{pmatrix},$$

$$f \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z^2 & 0 \end{pmatrix}.$$

It can be easily proved that d is a multiplicative derivation of $\mathcal{N}$ and f is a left generalized multiplicative derivation of $\mathcal{N}$ with an associated multiplicative derivation d of $\mathcal{N}$. But f is not a right generalized multiplicative derivation of $\mathcal{N}$ associated with multiplicative derivation d. It can be also verified that the maps d, f defined here are non-additive.

The study of commutativity of 3-prime near-rings was initiated by using derivations by H.E. Bell and G. Mason [6] in 1987. Subsequently a number of authors have investigated the commutativity of 3-prime near-rings admitting different types of derivations, generalized derivations, generalized multiplicative derivations(for reference see [1, 3, 4, 5, 6, 7, 8, 9], where further references can be found). In the present paper, we have obtained the commutativity of 3-prime near-rings, equipped with left generalized multiplicative derivations and satisfying some differential identities or conditions.

2. Preliminary Results

In this section we give some well-known results and we add some new lemmas which will be used throughout the next section of the paper. The proofs of the Lemmas 2.1 − 2.4 can be found in [6, 4], while those of Lemmas 2.5 − 2.7, can be found in [6, Lemma 1],[11, Lemma 2.1] and [14, Lemma 2] respectively.

Lemma 2.1. Let $\mathcal{N}$ be a 3-prime near-ring. If $Z \setminus \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.

Lemma 2.2. Let $\mathcal{N}$ be a 3-prime near-ring. If $z \in Z \setminus \{0\}$ and x is an element of $\mathcal{N}$ such that $xz \in Z$ or $zx \in Z$ then $x \in Z$.

Lemma 2.3. Let $\mathcal{N}$ be a 3-prime near-ring and $\mathfrak{A}$ be nonzero semigroup ideal of $\mathcal{N}$. Let d be a nonzero derivation on $\mathcal{N}$. If $x \in \mathcal{N}$ and $xd(\mathfrak{A}) = \{0\}$, then $x = 0$.

Lemma 2.4. Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d for which $d(\mathfrak{A}) \subseteq Z$, then $\mathcal{N}$ is a commutative ring.

Lemma 2.5. Let $\mathcal{N}$ be a near-ring and d be a derivation on $\mathcal{N}$. Then $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$ for all $x, y, z \in \mathcal{N}$.

Lemma 2.6. A near-ring $\mathcal{N}$ admits a multiplicative derivation if and only if it is zero-symmetric.

Lemma 2.7 Let $\mathcal{N}$ be a near-ring with center Z and let d be derivation on $\mathcal{N}$. Then $d(Z) \subseteq Z$.

Lemma 2.8. Let $\mathcal{N}$ be 3-prime near-ring. If $\mathcal{N}$ admits a left generalized multiplicative derivation f with associated multiplicative derivation d such that $f(u)v = uf(v)$ for all $u, v \in \mathcal{N}$, then $d = 0$.

Proof. We are given that $f(u)v = uf(v)$ for all $u, v \in \mathcal{N}$. Now replacing v by vw, where $w \in \mathcal{N}$, in the previous relation, we obtain that $f(u)vw = uf(vw)$ i.e.; $f(u)vw = u(vf(w) + d(v)w)$. By using hypothesis we arrive at $ud(v)w = 0$ i.e.; $u\mathcal{N}d(v)w = \{0\}$. Now using the facts that $\mathcal{N} \neq \{0\}$ and $\mathcal{N}$ is a 3-prime near-ring, we obtain that $d(v)w = 0$, for all $v, w \in \mathcal{N}$. This shows that $d(v)w = 0$ i.e.; $d(\mathcal{N})\mathcal{N}w = \{0\}$. Again 3-primeness of $\mathcal{N}$ and $\mathcal{N} \neq \{0\}$ force us to conclude that $d(\mathcal{N}) = \{0\}$. We get $d = 0$.

3. Main Results

We facilitate our discussion with the following theorem.

Theorem 3.1. Let f be a nonzero left generalized multiplicative derivation with associated nonzero multiplicative derivation d of a 3-prime near-ring $\mathcal{N}$ such that $f([x,y]) = 0$ for all $x, y \in \mathcal{N}$. Then $\mathcal{N}$ is a commutative ring.

Proof. Assume that $f([x,y]) = 0$ for all $x, y \in \mathcal{N}$. Putting xy in place of y, we obtain that $f([x,xy]) = f(x[x,y]) = xf([x,y]) + d(x)[x,y] = 0$. Using hypothesis, it is clear that

$$d(x)xy = d(x)yx \text{ for all } x, y \in \mathcal{N}. \tag{3.1}$$

Replacing y by yr where $r \in \mathcal{N}$ in (3.1) and using this relation again, we get $d(x)\mathcal{N}[x,r] = \{0\}$ for all $x, r \in \mathcal{N}$. Hence by 3-primeness of N, for each $x \in \mathcal{N}$ either $d(x) = 0$ or $x \in Z$. Let $u \in \mathcal{N}$. It is clear that either $d(u) = 0$ or $u \in Z$. We claim that if $d(u) = 0$, then also $u \in Z$. Suppose on contrary i.e.; $u \notin Z$. Now in the present situation, we prove that $d(uv) \neq 0$, for all $v \in \mathcal{N}$. For otherwise, we have $d(uv) = 0$ for all $v \in \mathcal{N}$, which gives us $ud(v) + d(u)v = 0$. This implies that $ud(v) = 0$ for all $v \in \mathcal{N}$. Replacing v by vr, where $r \in \mathcal{N}$, in the previous relation and using the same again, we arrive at $u\mathcal{N}d(r) = \{0\}$. Using the facts that $\mathcal{N}$ is 3-prime and $d \neq 0$, we obtain that $u = 0 \in Z$, which leads to a contradiction. Thus, we have seen that if $d(u) = 0$ and $u \notin Z$, then there exists $v \in \mathcal{N}$, such that $d(uv) \neq 0$ and obviously $v \neq 0$. Since $u, v \in \mathcal{N}$, we have $uv \in \mathcal{N}$. We obtain that either $d(uv) = 0$ or $uv \in Z$. But as $d(uv) \neq 0$, we infer that $uv \in Z$. Next we claim that $v \notin Z$, for otherwise we have $uvr = ruv$ i.e.; $v[u,r] = 0$ for all $r \in \mathcal{N}$. This shows that $v\mathcal{N}[u,r] = \{0\}$. Now by 3-primeness of $\mathcal{N}$, we conclude that $u \in Z$, as $v \neq 0$, leading to a contradiction. Including all the above arguments, we conclude that if $d(u) = 0$ and $u \notin Z$, then there exists $v \in \mathcal{N}$, such that $d(uv) \neq 0$ and $v \notin Z$. As $v \notin Z$, shows that $d(v) = 0$. Finally, we get $d(uv) = ud(v) + d(u)v = u0 + 0v = 0$, leading to a contradiction again. We have proved that if $d(u) = 0$, then also $u \in Z$ i.e.; $\mathcal{N} \subseteq Z$. Thus we obtain that $N = Z$ i.e; N is a commutative near-ring. If $N = \{0\}$ then N is trivially a commutative ring. If $N \neq \{0\}$ then there exists $0 \neq x \in N$ and hence $x+x \in N = Z$. Now by Lemma 2.1; we conclude that N is a commutative ring.

Theorem 3.2. Let f be a nonzero left generalized multiplicative derivation with associated nonzero multiplicative derivation d of $\mathcal{N}$ such that $f[x,y] = x^k[x,y]x^l$, k, l; being some given fixed positive integers, for all $x, y \in \mathcal{N}$. Then $\mathcal{N}$ is a commutative ring.

Proof. It is given that $f[x,y] = x^k[x,y]x^l$, for all $x, y \in \mathcal{N}$. Replacing y by xy in the previous relation, we obtain that $f[x,xy] = x^k[x,xy]x^l$, i.e.; $f(x[x,y]) = x^k(x[x,y])x^l$. This implies that $xf[x,y] + d(x)[x,y] = x^kx[x,y]x^l = xx^k[x,y]x^l = xf[x,y]$. Now we obtain that $d(x)[x,y] = 0$ for all $x, y \in \mathcal{N}$, which is same as the relation (3.1) of Theorem

3.1. Now arguing in the same way as in the Theorem 3.1., we conclude that $\mathcal{N}$ is a commutative ring.

The following example shows that the restriction of 3-primeness imposed on the hypotheses of Theorems 3.1 and 3.2 is not superfluous.

Example 3.2. Consider the near-ring $\mathcal{N}$, taken as in Example 1.1. $\mathcal{N}$ is not 3-prime and

(i) $f([x,y]) = 0$,

(ii) $f[x,y] = x^k[x,y]x^l$, k,l; being some given fixed positive integers, for all $x,y \in \mathcal{N}$. However $\mathcal{N}$ is not a commutative ring.

Theorem 3.3. Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero left generalized multiplicative derivation f with associated nonzero multiplicative derivation d such that either (i) $f([x,y]) = [f(x),y]$ for all $x,y \in \mathcal{N}$, or (ii) $f([x,y]) = [x,f(y)]$, for all $x,y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.

Proof. (i) Given that $f([x,y]) = [f(x),y]$, for all $x,y \in \mathcal{N}$. Replacing y by xy in the previous relation, we get $f([x,xy]) = [f(x),xy]$ i.e.; $f(x[x,y]) = [f(x),xy]$. This shows that $xf([x,y]) + d(x)[x,y] = f(x)xy - xyf(x)$. Using the given condition and the fact that $[f(x),x] = 0$, the previous relation reduces to $x(f(x)y - yf(x)) + d(x)[x,y] = xf(x)y - xyf(x)$. This gives us $d(x)[x,y] = 0$, for all $x,y \in \mathcal{N}$, which is the same as the relation (3.1) of Theorem 3.1. Now arguing in the similar way as in the Theorem 3.1., we conclude that $\mathcal{N}$ is a commutative ring.

(ii) We have $f([x,y]) = [x,f(y)]$, for all $x,y \in \mathcal{N}$. Replacing x by yx in the given condition, we obtain that $f([yx,y]) = [yx,f(y)]$ i.e.; $f(y[x,y]) = yxf(y) - f(y)yx$. This gives us $yf([x,y]) + d(y)[x,y] = yxf(y) - f(y)yx$. With the help of the given condition and using the fact that $[f(y),y] = 0$, previous relation reduces to $yxf(y) - yf(y)x + d(y)[x,y] = yxf(y) - yf(y)x$. As a result, we obtain that $d(y)[x,y] = 0$ i.e.; $d(y)[y,x] = 0$. This implies that $d(x)[x,y] = 0$, which is identical with the relation (3.1) of Theorem 3.1. Now arguing in the similar way as in the Theorem 3.1., we conclude that $\mathcal{N}$ is a commutative ring.

Theorem 3.4. Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero left generalized multiplicative derivation f with associated nonzero multiplicative derivation d such that either (i) $f([x,y]) = [d(x),y]$ for all $x,y \in \mathcal{N}$, or (ii) $d([x,y]) = [f(x),y]$, for all $x,y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.

Proof. (i) We are given that $f([x,y]) = [d(x),y]$. Replacing y by xy in the previous relation we get $f([x,xy]) = [d(x),xy]$. This relation gives $f(x[x,y]) = [d(x),xy]$ i.e.; $xf([x,y]) + d(x)[x,y] = d(x)xy - xyd(x)$. Using the given condition and the fact that $[d(x),x] = 0$, we obtain that $xd(x)y - xyd(x) + d(x)[x,y] = xd(x)y - xyd(x)$. Finally we get $d(x)[x,y] = 0$, for all $x,y \in \mathcal{N}$, which is the same as the relation (3.1) of Theorem 3.1. Now arguing in the similar way as in the Theorem 3.1., we conclude that $\mathcal{N}$ is a commutative ring.

(ii) We have $d([x,y]) = [f(x),y]$. Putting xy in the place of y in the previous relation we get $d([x,xy]) = [f(x),xy]$. This implies that $d(x[x,y]) = f(x)xy - xyf(x)$ i.e.; $xd[x,y] + d(x)[x,y] = f(x)xy - xyf(x)$. Using the fact that $[f(x),x] = 0$, we obtain that $xd[x,y] + d(x)[x,y] = xf(x)y - xyf(x)$ i.e.; $xd[x,y] + d(x)[x,y] = x[f(x),y]$. Now using the hypothesis, we get $xd[x,y] + d(x)[x,y] = xd([x,y])$. Finally we have $d(x)[x,y] = 0$, for all $x,y \in \mathcal{N}$, which is identical with the relation (3.1) of Theorem 3.1. Now arguing in the similar way as in the Theorem 3.1., we conclude that $\mathcal{N}$ is a commutative ring.

Theorem 3.5. Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero left generalized multiplicative derivation f with associated nonzero multiplicative derivation d such that either (i) $f([x,y]) = \pm[x,y]$ for all $x,y \in \mathcal{N}$, or (ii) $f([x,y]) = \pm(xoy)$ for all $x,y \in \mathcal{N}$, then under the condition (i) $\mathcal{N}$ is a commutative ring and under the condition (ii) $\mathcal{N}$ is a commutative ring of characteristic 2.

Proof. Assume that condition (i) holds i.e.; $f([x,y]) = \pm[x,y]$ for all $x,y \in \mathcal{N}$. Putting xy in place of y, we obtain, $f([x,xy]) = f(x[x,y]) = xf([x,y]) + d(x)[x,y] = \pm x[x,y]$. Using our hypothesis we get $d(x)xy = d(x)yx$ for all $x,y \in \mathcal{N}$, which is identical with the relation (3.1) of Theorem 3.1. Now arguing in the similar way as in the Theorem 3.1., we conclude that $\mathcal{N}$ is a commutative ring. Under the condition (ii), using similar arguments, it is easy to show that N a commutative ring. But now under this situation, condition (ii) reduces to $xoy = 0$ for all $x,y \in \mathcal{N}$ i.e.; $2xy = 0$. Suppose on contrary i.e.; characteristic $\mathcal{N} \neq 2$. As $\mathcal{N}$ is a prime ring, $\mathcal{N}$ will be a 2-torsion free ring. Now we get $xy = 0$, for all $x,y \in \mathcal{N}$ i.e.; $x\mathcal{N}y = \{0\}$. Finally, we have $\mathcal{N} = \{0\}$, leading to a contradiction.

Theorem 3.6. Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero left generalized multiplicative derivation f with associated multiplicative derivation d such that $f(xy) = \pm(xy)$ for all $x,y \in \mathcal{N}$, then $d = 0$.

Proof. Let $f(xy) = xy$ for all $x,y \in \mathcal{N}$. Putting yz, where $z \in \mathcal{N}$ for y in the previous relation, we obtain that $f(x(yz)) = x(yz)$ i.e.; $xf(yz) + d(x)yz = xyz$. Using the hypothesis we get $d(x)yz = 0$ i.e.; $d(x)\mathcal{N}z = \{0\}$. Since $\mathcal{N} \neq \{0\}$, by 3-primeness of N, we get $d = 0$. Similar arguments hold if $f(xy) = -(xy)$ for all $x, y \in \mathcal{N}$.

Very recently, Boua and Kamal [7, Theorem 1] proved that if $\mathcal{N}$ is a 3-prime near-ring, which admits nonzero derivations d_1 and d_2 such that $d_1(x)d_2(y) \in Z$, for all $x, y \in \mathfrak{A}$, where $\mathfrak{A}$ is a nonzero semigroup ideal of $\mathcal{N}$, then $\mathcal{N}$ is a commutative ring. Motivated by this result, we have obtained the following:

Theorem 3.7. Let $\mathcal{N}$ be a 3-prime near-ring and f be a nonzero left generalized multiplicative derivation with associated nonzero multiplicative derivation d' of $\mathcal{N}$ such that either (i) $f(x)d(y) \in Z$, for all $x, y \in \mathcal{N}$, or (ii) $d(x)f(y) \in Z$, for all $x, y \in \mathcal{N}$ and d is a nonzero derivation of $\mathcal{N}$. Then $\mathcal{N}$ is a commutative ring.

Proof. (i) We are given that $f(x)d(y) \in Z$, for all $x, y \in \mathcal{N}$. Replacing y by yz, where $z \in \mathcal{N}$ in the previous relation, we get $f(x)d(yz) \in Z$. This implies that $f(x)(yd(z) + d(y)z) \in Z$ i.e.; $f(x)yd(z) + f(x)d(y)z \in Z$. This gives us $(f(x)yd(z) + f(x)d(y)z)z = z(f(x)yd(z) + f(x)d(y)z)$. Now using Lemma 2.5, we obtain that $f(x)yd(z)z + f(x)d(y)zz = zf(x)yd(z) + zf(x)d(y)z$ for all $x, y, z \in \mathcal{N}$. Using the hypothesis we infer that $f(x)yd(z)z + f(x)d(y)z^2 = zf(x)yd(z) + f(x)d(y)z^2$ i.e.; $f(x)yd(z)z = zf(x)yd(z)$. Putting $d(t)y$ for y, where $t \in \mathcal{N}$ in the relation $f(x)yd(z)z = zf(x)yd(z)$, we get $f(x)d(t)yd(z)z = zf(x)d(t)yd(z)$ and now using the hypothesis again, we arrive at $f(x)d(t)(yd(z)z - zyd(z))$. This shows that $f(x)d(t)\mathcal{N}(yd(z)z - zyd(z)) = \{0\}$. Hence 3-primeness of $\mathcal{N}$ shows that either $f(x)d(t) = 0$ or $yd(z)z - zyd(z) = 0$. We claim that $f(x)d(t) \neq 0$ for all $x, t \in \mathcal{N}$. For otherwise if $f(x)d(t) = 0$ for all $x, t \in \mathcal{N}$, we have $f(x)d(\mathcal{N}) = \{0\}$. Using Lemma 2.3, we find that $f(x) = 0$ for all $x \in \mathcal{N}$, leading to a contradiction. Thus there exist $x_0, t_0 \in \mathcal{N}$ such that $f(x_0)d(t_0) \neq 0$. Hence, we arrive at $yd(z)z - zyd(z) = 0$ for all $y, z \in \mathcal{N}$. Now replacing y by $yf(x)$, where $x \in \mathcal{N}$ in the previous relation and using the hypothesis again, we get $f(x)d(z)(yz - zy) = 0$ i.e.; $f(x)d(z)\mathcal{N}(yz - zy) = \{0\}$. By hypothesis we have $f(\mathcal{N}) \neq \{0\}$, hence there exists $u_0 \in \mathcal{N}$ such that $f(u_0) \neq 0$. By Lemma 2.3, there exists $z_0 \in \mathcal{N}$ such that $f(u_0)d(z_0) \neq 0$ and hence obviously $d(z_0) \neq 0$. Again 3-primeness of $\mathcal{N}$ and the relation $f(x)d(z)\mathcal{N}(yz - zy) = \{0\}$, ultimately give us $yz_0 = z_0y$ for all $y \in \mathcal{N}$. Now Lemma 2.5 insures that $z_0 \in Z$ and using Lemma 2.7, we obtain that $d(z_0) \in Z$. Since $f(u_0)d(z_0) \in Z$ and $0 \neq d(z_0) \in Z$, Lemma 2.2, implies that $f(u_0) \in Z$. Using the given hypothesis again we have $f(u_0)d(y) \in Z$. But $0 \neq f(u_0) \in Z$, thus Lemma 2.2, shows that $d(\mathcal{N}) \subseteq Z$. Finally the Lemma 2.4, gives the required result.

(ii) Using the similar arguments as used in (i) with necessary variations, it can be easily shown that under the condition $d(x)f(y) \in Z$, for all $x, y \in \mathcal{N}$, $\mathcal{N}$ is a commutative ring.

Theorem 3.8**.** Let $\mathcal{N}$ be a 3-prime near-ring and f be a left generalized multiplicative derivation of $\mathcal{N}$ such that either (i) $d(y)f(x) = [x, y]$, for all $x, y \in \mathcal{N}$, or (ii) $d(y)f(x) = -[x, y]$, for all $x, y \in \mathcal{N}$ and d is a nonzero derivation of $\mathcal{N}$. Then $\mathcal{N}$ is a commutative ring.

Proof. (i) We are given that

$$d(y)f(x) = [x, y], \text{ for all } x, y \in \mathcal{N}. \tag{3.2}$$

Case I: Let $f = 0$. Under this condition the equation (3.2) reduces to $[x, y] = 0$ for all $x, y \in \mathcal{N}$. This implies that $xy = yx$, for all $x, y \in \mathcal{N}$. Replacing x by xr, where $r \in \mathcal{N}$ in the previous relation and using the same relation again we arrive at $\mathcal{N}[r, y] = \{0\}$ i.e.; $[r, y]\mathcal{N}[r, y] = \{0\}$. Now using 3-primeness of N, we conclude that $r \in Z$. This implies that $\mathcal{N} \subseteq Z$. If $N = \{0\}$, then N is trivially a commutative ring. If $N \neq \{0\}$ then there exists $0 \neq x \in N$ and hence $x + x \in N = Z$. Now by Lemma 2.1; we conclude that N is a commutative ring.

Case II: Let $f \neq 0$. Replacing y by xy in the relation (3.2), we obtain that $d(xy)f(x) = x[x, y]$ i.e.; $(xd(y) + d(x)y)f(x) = x[x, y]$. Using Lemma 2.5 and the relation (3.2), we arrive at $xd(y)f(x) + d(x)yf(x) = xd(y)f(x)$. This shows that $d(x)yf(x) = 0$ i.e.; $d(x)\mathcal{N}f(x) = \{0\}$. Using 3-primeness of N, we conclude that for any given $x \in \mathcal{N}$, either $d(x) = 0$ or $f(x) = 0$. If for any given $x \in \mathcal{N}$, $f(x) = 0$, then relation (3.2) reduces to $[x, y] = 0$ for all $y \in \mathcal{N}$ i.e.; $x \in Z$. By Lemma 2.7, this shows that $d(x) \in Z$. Finally using both possibilities, we deduce that $d(\mathfrak{A}) \subseteq Z$. By Lemma 2.4, we get our required result.

(ii) Using the similar arguments as used in (i), it can be easily proved that under the condition $d(y)f(x) = -[x, y]$, for all $x, y \in \mathcal{N}$, $\mathcal{N}$ is a commutative ring.

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