An Efficient Product Estimator using Harmonic Mean

K. B. Panda, N. Sahoo
Utkal University, Vani Vihar, Odisha, India.

Abstract: *In this paper a new product estimator has been proposed by exploiting the product estimators due to Srivastava (1983), Agrawal and Jain (1989) and Panda and Sahoo (2015) . The expressions of the bias and mean square error of the proposed estimator, to the first order of approximation, are derived in general form. The new product estimator is found to perform better than its competing estimators from the standpoint of bias and mean square error both in one-phase sampling and two-phase sampling under conditions which hold good in practice. The theoretical findings are supported by a numerical illustration.*

Key words: *Auxiliary variable, Product estimator, Bias and Mean squared error, Predictive estimation*

1. Introduction

In survey sampling we consider a population of size N whose units are arbitrarily labelled $1,2,\ldots\ldots,N$ and let y_i and x_i be the values for the i^{th} unit $(i=1,2,\ldots.N)$ in respect of the study variable y and the auxiliary variable x, respectively. With a view to estimating the population total $Y = y_1 + y_2 + \cdots + y_N$ or the population mean $\bar{Y} = \frac{Y}{N}$, a sample of size n drawn by simple random sampling without replacement. Under the assumption that y and x are negatively correlated, a possible choice for estimating the population mean $\bar{Y}$ is the customary product estimator given by

$$\bar{y}_P = \frac{\bar{y}\bar{x}}{\bar{X}}, \tag{1.1}$$

Where $\bar{y}$ and $\bar{x}$ are, respectively, the sample means in respect of the study and the auxiliary variables and $\bar{X}$ is the population mean of the auxiliary variable.

Making use of (1.1) as the mean per unit for the unobserved units in the population, Srivastava (1983) invoked the usual predictive approach due to Basu (1971) to suggest the estimator

$$\bar{y}_P' = \frac{n\bar{y}}{N} + \frac{(\mathrm{N-n})^2\bar{y}\bar{x}}{N(N\bar{X}-\mathrm{n}\bar{x})}. \tag{1.2}$$

Agrawal and Jain (1989) proposed a predictive product estimator given by

$$\bar{y}_P'' = \frac{\bar{y}\tilde{x}}{\tilde{X}}, \tag{1.3}$$

where $\tilde{x}$ and $\tilde{X}$ are, respectively, the sample and population harmonic means of x values defined as

$$\tilde{x} = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} \text{ and } \tilde{X} = \frac{N}{\sum_{i=1}^{N}\frac{1}{x_i}}.$$

Combining the ideas due to Srivastava (1983) and Agrawal and Jain (1989), Panda and Sahoo (2015) proposed the product estimator

$$\bar{y}_P''' = \frac{(\mathrm{N-n})\bar{y}\tilde{x}}{(\mathrm{N}\tilde{X}-\mathrm{n}\tilde{x})}. \tag{1.4}$$

2. The new product estimator

we propose the following product estimator

$$\bar{y}_{HP} = \bar{y}\left(\frac{\bar{x}^{1/2}+\tilde{x}^{1/2}}{\bar{X}^{1/2}+\tilde{X}^{1/2}}\right)^{\delta}, \tag{2.1}$$

where the symbols have their usual meanings. The bias and mean square error of the estimator to the 1^{st} degree of approximation, i.e., to $O(n^{-1})$ are given, respectively, by

$$B(\bar{y}_{HP}) = \frac{\mathrm{N-n}}{\mathrm{Nn}}\bar{Y}\left(\frac{\delta}{2}\rho C_y C_x + \frac{\delta^2}{8}C_x^2\right) \tag{2.2}$$

$$MSE(\bar{y}_{HP}) = \frac{\mathrm{N-n}}{\mathrm{Nn}}\bar{Y}^2\left(C_y^2 + \delta\rho C_y C_x + \frac{\delta^2}{4}C_x^2\right), \tag{2.3}$$

here $C_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(y_i-\bar{Y})^2}{\bar{Y}^2}$, $C_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(x_i-\bar{X})^2}{\bar{X}^2}$ and ρ is the correlation coefficient between y and x, assumed to be negative.

3. Biases and Mean square errors of the competing estimators

The biases of the competing estimators to $O(n^{-1})$ have been arrived at as follows:

$$B(\bar{y}_P) = \frac{\mathrm{N-n}}{\mathrm{Nn}}\bar{Y}\rho C_y C_x \tag{3.1}$$

$$B(\bar{y}_P') = \frac{\mathrm{N-n}}{\mathrm{Nn}}\bar{Y}\left(\rho C_y C_x + \frac{n}{N-n}C_x^2\right) \tag{3.2}$$

$$B(\bar{y}_P'') = \frac{\mathrm{N-n}}{\mathrm{Nn}}\bar{Y}\left(\rho C_y C_x + C_x^2\right) \tag{3.3}$$

$$B(\bar{y}_P''') = \frac{\mathrm{N-n}}{\mathrm{Nn}}\bar{Y}\left(\frac{\mathrm{N}}{\mathrm{N-n}}\rho C_y C_x + \frac{\mathrm{N}^2}{(N-n)^2}C_x^2\right). \tag{3.4}$$

Again the variance or mean square error of mean per unit estimator is given by

$$V(\bar{y}) = MSE(\bar{y}) = \frac{(N-n)}{Nn}\bar{Y}^2 C_y^2 \tag{3.5}$$

The mean square errors of the competing estimators $\bar{y}_P, \bar{y}_{P_1}, \bar{y}_P', \bar{y}_P''$ are, to $O(n^{-1})$, found to be same and is as follows:

$$MSE(\bar{y}_P) = MSE(\bar{y}_{P_1}) = MSE(\bar{y}_P^{'}) = MSE(\bar{y}_P^{''})$$

$$= \frac{N-n}{Nn}\bar{Y}^2(C_y^2 + 2\rho C_y C_x + C_x^2), \quad (3.6)$$

and $MSE(\bar{y}_P^{'''}) = \frac{N-n}{Nn}\bar{Y}^2\left(C_y^2 + \frac{2N}{N-n}\rho C_y C_x + \frac{N^2}{(N-n)^2}C_x^2\right). \quad (3.7)$

4. Optimum choice of the scalar δ

Differentiating (2.3) with respect to δ and equating it to zero, we get the optimum value of δ as

$$\delta_{opt.} = -2\rho\frac{C_y}{C_x} \quad (4.1)$$

5. Efficiency Comparison

In this section, we have derived the conditions under which the proposed estimator $\bar{y}_{HP}$ is more efficient than the estimators $\bar{y}_P, \bar{y}_{P_1}, \bar{y}_P^{'}, \bar{y}_P^{''}$ and $\bar{y}_P^{'''}$. From (2.3), (3.5), (3.6) and (3.7) we find

$$MSE(\bar{y}_{HP}) < MSE(\bar{y})$$

$$\text{if } \rho\frac{C_y}{C_x} < 0. \quad (5.1)$$

$$MSE(\bar{y}_{HP}) < MSE(\bar{y}_P) = MSE(\bar{y}_{P_1}) = MSE(\bar{y}_P^{'}) = MSE(\bar{y}_P^{''})$$

$$\text{if } \rho\frac{C_y}{C_x} < -1 \quad (5.2)$$

$$MSE(\bar{y}_{HP}) < MSE(\bar{y}_P^{'''})$$

$$\text{if } \rho\frac{C_y}{C_x}\left(\rho\frac{C_y}{C_x} + \frac{2N}{N-n}\right) > -\frac{N^2}{(N-n)^2} \quad (5.3)$$

6. Biases and mean square errors in two- phase sampling

When $\bar{X}$ is not known, we take recourse to two-phase sampling or double sampling. Under the technique, the expressions for the biases and mean square errors of the four estimators to the 1^{st} degree of approximation, i.e, to $O(n^{-1})$ are as given below:

$$B(\bar{y}_{Pd}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\bar{Y}\rho C_y C_x \quad (6.1)$$

$$B(\bar{y}_{Pd}^{'}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\bar{Y}\left(\rho C_y C_x + \frac{n}{N-n}C_x^2\right) \quad (6.2)$$

$$B(\bar{y}_{Pd}^{''}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\bar{Y}(\rho C_y C_x + C_x^2) \quad (6.3)$$

$$B(\bar{y}_{Pd}^{'''}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\bar{Y}\left(\frac{N}{N-n}\rho C_y C_x + \frac{N^2}{(N-n)^2}C_x^2\right) \quad (6.4)$$

$$B(\bar{y}_{HPd}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\bar{Y}\left(\frac{\delta}{2}\rho C_y C_x + \frac{\delta^2}{8}C_x^2\right) \quad (6.5)$$

$$MSE(\bar{y}_{Pd}) = MSE(\bar{y}_{P_{1d}}) = MSE(\bar{y}_{Pd}^{'}) = MSE(\bar{y}_{Pd}^{''})$$

$$= \bar{Y}^2\left[\left(\frac{1}{n} - \frac{1}{N}\right)C_y^2 + \left(\frac{1}{n} - \frac{1}{n'}\right)(2\rho C_y C_x + C_x^2)\right] \quad (6.6)$$

and $MSE(\bar{y}_{Pd}^{'''}) =$

$$\bar{Y}^2\left[\left(\frac{1}{n} - \frac{1}{N}\right)C_y^2 + \left(\frac{1}{n} - \frac{1}{n'}\right)\left\{\frac{2N}{N-n}\rho C_y C_x + \frac{N^2}{(N-n)^2}C_x^2\right\}\right]. \quad (6.7)$$

$MSE(\bar{y}_{HPd}) =$

$$\bar{Y}^2\left[\left(\frac{1}{n} - \frac{1}{N}\right)C_y^2 + \left(\frac{1}{n} - \frac{1}{n'}\right)\left\{\delta\rho C_y C_x + \frac{\delta^2}{4}C_x^2\right\}\right]. \quad (6.8)$$

Comparison of biases and mean square errors yield the same results as in the case of one-phase sampling discussed earlier.

7. Numerical illustrations

Illustration 1

For the purpose of establishing the superiority of $\bar{y}_{HP}$ over its competing estimators, we refer to an example from Maddala (1977, p. 96).

Y represents per-capita consumption, which is negatively correlated with deflated price of meat, i.e. beef (x). The following quantities have been computed:

$$N = 16, \quad n = 4, \bar{Y} = 72.625, \bar{X} = 77.362, C_y = 0.129, \; C_x = 0.152 \text{ and } \rho = -0.844$$

Table-7.1

Sl.No.	Estimators	Absolute biases	Mean square error	Percentage gain in efficiency over $\bar{y}$
1	$\bar{y}$	0	16.4571	0
2	$\bar{y}_{P_1}$	0	6.5270	152.1388
3	$\bar{y}_P$	0.2253	6.5270	152.1388
4	$\bar{y}_P^{'}$	0.1198	6.5270	152.1388
5	$\bar{y}_P^{''}$	0.0899	6.5270	152.1388
6	$\bar{y}_P^{'''}$	0.2596	13.4497	22.3603
7	$\boldsymbol{\bar{y}_{HP}}$	**0.0799**	**4.6480**	**254.0684**

Illustration 2

Johnston, page 171

y = Percentage of hives affected by disease

x = Date of flowering of a particular summer species (number of days from January 1)

$$N = 10, \; n = 4, \; \bar{Y} = 52, \bar{X} = 200, C_y = 0.1562, \; C_x = 0.04583 \;\; and \; \rho = -0.94$$

Table 7.2

Sl.No.	Estimators	Absolute biases	Mean square error	Percentage gain in efficiency over $\bar{y}$
1	$\bar{y}$	0	9.8960	0
2	$\bar{y}_{P_1}$	0	5.3134	86.25
3	$\bar{y}_P$	0.0525	5.3134	86.25
4	$\bar{y}_P'$	0.0413	5.3134	86.25
5	$\bar{y}_P''$	0.0359	5.3134	86.25
6	$\bar{y}_P'''$	0.0421	3.1637	212.79
7	$\boldsymbol{\bar{y}_{HP}}$	**0.0842**	**1.1762**	**741.35**

The above table clearly points to the fact that the proposed estimator performs better than its competing estimators.

Acknowledgements

The authors are thankful to the editor and learned referee for their valuable suggestions regarding the improvement of this paper.

REFERENCES

1. Adewara, A.A.,Singh, Rajesh and Mukesh Kumar (2012): "Efficiency of some modified ratio and product estimators using known value of some population parameters", International journal of applied science and technology, Vol.2, No.2, 76-79.
2. Agrawal, M.C. and Jain, N. (1989): " A new predictive product estimator ", Biometrika, 76(4), 822-823.
3. Basu, D. (1971): " An essay on the logical foundations of survey sampling, Part I, Foundations of statistical inference, Ed. By V.P. Godambe and D.A. Sprott, New York 1971,203-233.
4. Maddala, G.S. (1977): " Econometrics", McGraw Hills Pub. Co., New York.
5. Panda, K.B. and Sahoo, N. (2015): "Study on product estimation", International journal of multidisciplinary research review, Vol.1,Issue –10, Dec -2015. Page 165-167.
6. Srivastava, S.K. (1983): "Predictive estimation of finite population mean using product estimator", Metrika, 30, 93-99.
7. Sukhatme, P.V., Sukhatme, B.V., Sukhatme,S. and Asok, C.(1984): ''Sampling theory of surveys with applications'', Iowa State University press, USA.