Super Edge Graceful and Even Edge Graceful Labeling of Cayley Digraphs

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Abstract —*In this paper we define super edge graceful labeling and even edge graceful labeling for digraphs and we show that the Cayley digraph admits super edge graceful labeling and even edge graceful labeling.*
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I. INTRODUCTION

One of the main topics in graph theory is labeling the graph which was introduced by Rosa in 1965 [1]. A graph labeling is an assignment of integers to edges or vertices or both. The labeling technology has many applications in real life such as Radar, Astronomy, X-Ray, and communication system.

Rosa [2] introduced the notion of graceful labeling. Bloom and Hsu [3],[4],[5] extended the concept of graceful labeling for digraphs also. Lo [6] introduced the idea of edge graceful graphs. Lee [7] further developed the concept for more graphs. Lee et al [8] defined super edge graceful labeling for some undirected graphs. Also Gayathri et al [9] discussed the even edge graceful labeling for undirected graphs. The Cayley graph that presents the exact structure of a group and this definition is introduced by Cayley [10]. Thirusangu et al [11] defined labeling concepts for some classes of Cayley digraphs. Further many authors started various labeling technologies on the Cayley digraphs [12],[13]. We now extended super edge graceful and the even edge graceful labeling to directed graphs and applied both the methodologies on the Cayley digraphs.

We can model the best interconnection network by graphs or digraphs according to the path (one way or two ways) between the nodes (vertices).The vertex transitivity is highly desirable for selecting the digraph to model the interconnection network, since the algorithm applied on one vertex can be applied on all vertices. Also the best model should have the property to reduce the implementation complexity and to increase the performance with constant cost. Keeping these scales, we can choose Cayley digraphs which provide the optimum solution for the above requirements.

II. PRELIMINARIES

In his division we have some basic definitions

A. *Definition*

A digraph G (V,E) with p vertices and q arcs is said to admit *super edge graceful labeling* if there is a bijection f from E to $\{0,\pm1,\pm2,\dots,\pm(q-1)/2\}$ when q is odd and from E to $\{\pm1,\pm2,\dots,\pm q/2\}$ when q is even such that the induced vertex labeling f* defined by $f^*(v_i) = \sum f(e_{ij})$ taken over the outgoing arcs of v_i, $1\le i \le p$ is a bijection from V to $\{0,\pm1,\pm2,\dots,\pm(p-1)/2\}$ when p is odd and from V to $\{\pm1,\pm2,\dots,\pm p/2\}$ when p is even.

B. *Definition*

A G(p,q) digraph with p vertices and q arcs $q \ge p$ is *even edge graceful* if there is an injection f from E to $\{1,2,3,\dots,2q\}$ such that the values of the induced mapping f* from V to $\{0,1,2,\dots,2q-1\}$ given by $f^*(v_i) = \sum f(e_{ij})$ (mod 2q) over all outgoing arcs of v_i, $1\le i \le p$ are even and distinct.

C. *Definition*

Let G be a finite group and let S be a generating subset of G. The *Cayley digraph* Cay (G,S) is the digraph whose vertices are the elements of G, and there is an arc from g to gs whenever g∈G and s∈S. If $S=S^{-1}$ then there is an arc from g to gs if and only if there is an arc from gs to g.

III. MAIN RESULTS

In this section we show that the Cayley digraph is super edge graceful and even edge graceful.

A. *Theorem*

The Cayley digraph Cay (G,S) is super edge graceful.

Proof:

Consider the Cayley digraph Cay (G,S) with p vertices and m number of generators, which has mp arcs totally and every vertex of the digraph contains m outgoing and m incoming arcs. Denote the vertex set of Cay (G,S) as

$V = \{v_1,v_2,\dots,v_p\}$ and the edge set of Cay (G,S) as

$E = E_{s_1} \cup E_{s_2} \cup \ldots, \cup E_{s_m}$
$= \{e_{11}, e_{12}, \ldots, e_{1m}, e_{21}\ e_{22}, \ldots, e_{2m}, \ldots, e_{p1}, \ldots, e_{pm}\}$
Where e_{ij} is an outgoing arc from i^{th} vertex generated by the generator s_j and
E_{s_1} = set of all outgoing arcs from v_i, $1 \le i \le p$ generated by s_1
E_{s_2} = set of all outgoing arcs from v_i, $1 \le i \le p$ generated by s_2

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E_{s_m} = set of all outgoing arcs from v_i, $1 \le i \le p$ generated by s_m

We prove the theorem in four cases according to the number of vertices and generators.

Case (i): If m and p are even

If m and n are even, then both the number of vertices and the number of edges are also even. To prove the theorem, we have to prove that there is a bijection f: E $\to$ $\{\pm 1, \pm 2, \ldots, \pm mp/2\}$ such that the induced function, $f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$, $1 \le i \le p$ is a bijection from V to $\{\pm 1, \pm 2, \ldots, \pm p/2\}$ where the summation is taken over the outgoing arcs of v_i.

Define f: E $\to$ $\{\pm 1, \pm 2, \ldots \pm mp/2\}$ as follows

For $1 \le i \le p$ and $1 \le j \le m-1$

$$f(e_{ij}) = \begin{cases} \frac{mp}{2} - i - \frac{p(j-1)}{2} + 1 & \text{for } j = 1,3,\ldots,m-1 \\ -[\frac{mp}{2} - i - \frac{p(j-2)}{2} + 1 & \text{for } j = 2,4,\ldots,m-2 \end{cases}$$

For $j = m$ and $1 \le i \le p$

$$f(e_{ij}) = \begin{cases} 2i - p - 1 & \text{for } 1 \le i \le \frac{p}{2} \\ 2i - 2p - 2 & \text{for } \frac{n}{2} + 1 \le i \le p \end{cases}$$

Then the induced function

$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$ for $1 \le i \le p$

$$= \begin{cases} \frac{mp}{2} - i + 1 - \left(\frac{mp}{2} - i + 1\right) + \frac{mp}{2} - i - p + 1 - \left(\frac{mp}{2} - i - p + 1\right) + \cdots + \frac{mp}{2} - i - \frac{(m-2)p}{2} + 1 + 2i - p - 1 & \text{for } 1 \le i \le \frac{p}{2} \\ \frac{mp}{2} - i + 1 - \left(\frac{mp}{2} - i + 1\right) + \frac{mp}{2} - i - p + 1 - \left(\frac{mp}{2} - i - p + 1\right) + \cdots + \frac{mp}{2} - i - \frac{(m-2)p}{2} + 1 + 2i - 2p - 2 & \text{for } \frac{p}{2} + 1 \le i \le p \end{cases}$$

$$= \begin{cases} i & \text{for } 1 \le i \le \frac{p}{2} \\ i - p - 1 & \text{for } \frac{p}{2} + 1 \le i \le p \end{cases}$$

Hence $f^*(V) = \{\pm 1, \pm 2, \ldots, \pm p/2\}$
For $1 \le s, t \le \frac{p}{2}$, when $s \ne t$, clearly $f^*(v_s) \ne f^*(v_t)$ and also
For $\frac{p}{2} + 1 \le s, t \le p$, when $s \ne t$, clearly $f^*(v_s) \ne f^*(v_t)$.
But for $1 \le s \le \frac{p}{2}$ and $\frac{p}{2} + 1 \le t \le p$,
when $s \ne t$, if $f^*(v_s) = f^*(v_t)$ then $s = t-p-1$
i.e., $p+1 = t-s$ which is a contradiction to $1 \le s, t \le p$
Therefore $f^*(v_s) \ne f^*(v_t)$.
Hence $f^*(V) = \{\pm 1, \pm 2, \ldots, \pm p/2\}$ is a bijection.
Therefore The Cayley digraph Cay (G,S) is super edge graceful when m and p are even.

Case (ii): If m is even and p is odd

If m is even and p is odd, then the number of vertices is odd and the number of edges is even. To prove the theorem, we have to prove that there is a bijection f: E $\to$ $\{\pm 1, \pm 2, \ldots, \pm mp/2\}$ such that the induced function, $f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$, $1 \le i \le p$ is a bijection from V $\to$ $\{0, \pm 1, \pm 2, \ldots, \pm (p-1)/2\}$ wherethe summation is taken over the outgoing arcs of v_i.

Define f: E $\to$ $\{\pm 1, \pm 2, \ldots, \pm mp/2\}$ as follows

For $1 \le i \le p$ and $1 \le j \le m-1$

$$f(e_{ij}) = \begin{cases} \frac{mp}{2} - i - \frac{p(j-1)}{2} + 1 & \text{for } j = 1,3,\ldots,m-1 \\ -[\frac{mp}{2} - i - \frac{p(j-2)}{2} + 1 & \text{for } j = 2,4,\ldots,m-2 \end{cases}$$

For $j = m$ and $1 \le i \le p$

$$f(e_{ij}) = \begin{cases} 2i-p-1 & \text{for } 1 \le i \le \frac{p-1}{2} \\ 2i-2p-2 & \text{for } \frac{p+1}{2} \le i \le n \end{cases}$$

Then the induced function,

$f^*(vi) = \sum_{j=1}^{m} f(e_{ij})$ for $1 \le i \le p$

$$= \begin{cases} \frac{mp}{2} - i + 1 - \left(\frac{mp}{2} - i + 1\right) + \frac{mp}{2} - i - p + 1 - \left(\frac{mp}{2} - i - p + 1\right) + \cdots + \frac{mp}{2} - i - \frac{(m-2)p}{2} + 1 + 2i - p - 1 & \text{for } 1 \le i \le \frac{p-1}{2} \\ \frac{mp}{2} - i + 1 - \left(\frac{mp}{2} - i + 1\right) + \frac{mp}{2} - i - p + 1 - \left(\frac{mp}{2} - i - p + 1\right) + \cdots + \frac{mp}{2} - i - \frac{(m-2)p}{2} + 1 + 2i - 2p - 1 & \text{for } \frac{p+1}{2} \le i \le p \end{cases}$$

$$= \begin{cases} i & for\ 1 \le i \le \frac{p-1}{2} \\ i-p & for\ \frac{p+1}{2} \le i \le p \end{cases}$$

Hence f* (V) = $\{0, \pm 1, \pm 2, \ldots, \pm (p-1)/2\}$

For $1 \le s,t \le \frac{p-1}{2}$, when $s \neq t$, clearly $f^*(v_s) \neq f^*(v_t)$ and also

For $\frac{p+1}{2} \le s,t \le p$, when $s \neq t$, clearly $f^*(v_s) \neq f^*(v_t)$.

But for $1 \le s \le \frac{p-1}{2}$ and $\frac{p+1}{2} \le t \le p$,

When $s \neq t$, if $f^*(v_s) = f^*(v_t)$ then s = t-p

i.e., p = t-s which is a contradiction to $1 \le s,t \le p$.

Therefore $f^*(v_s) \neq f^*(v_t)$.

Hence f* (V) = $\{0, \pm 1, \pm 2, \ldots, \pm (p-1)/2\}$is a bijection.

Therefore The Cayley digraph Cay (G,S) is super edge graceful when m is even and p is odd.

Case (iii): If m is odd and p is even

Though m is odd and p is even, both the number of vertices and the number of edges are even. To prove the theorem, we have to prove that there is a bijection f: E $\rightarrow \{\pm 1, \pm 2, \ldots, \pm mp/2\}$ such that the induced function, $f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$, $1 \le i \le p$ is a bijection from V to $\{\pm 1, \pm 2, \ldots, \pm p/2\}$ where the summation is taken over the outgoing arcs of v_i.

Define f : E $\rightarrow$ {±1,±2,…,±mp/2} as follows

For $1 \le i \le p$ and $1 \le j \le m-1$

$f(e_{ij})=$

$$\begin{cases} \frac{mp}{2} - i - \frac{p(j-1)}{2} + 1 & for\ j = 1,3,\ldots,m-2 \\ -[\frac{mp}{2} - i - \frac{p(j-2)}{2} + 1 & for\ j = 2,4,\ldots,m-1 \end{cases}$$

For j = m and $1 \le i \le p$

$$f(e_{ij}) = \begin{cases} \frac{p}{2} - i + 1 & for\ 1 \le i \le \frac{p}{2} \\ \frac{p}{2} - i & for\ \frac{p}{2} + 1 \le i \le p \end{cases}$$

Then the induced function,

$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$ for $1 \le i \le p$

$$= \begin{cases} \frac{mp}{2} - i + 1 - \left(\frac{mp}{2} - i + 1\right) + \frac{mp}{2} - i - p + 1 - \left(\frac{mp}{2} - i - p + 1\right) + \cdots + \frac{mp}{2} - i - \frac{(m-3)p}{2} + 1 - \left(\frac{mp}{2} - i - \frac{(m-3)p}{2} + 1\right) + \frac{p}{2} - i + 1 & \text{for } 1 \le i \le \frac{p}{2} \\ \frac{mp}{2} - i + 1 - \left(\frac{mp}{2} - i + 1\right) + \frac{mp}{2} - i - p + 1 - \left(\frac{mp}{2} - i - p + 1\right) + \cdots + \frac{mp}{2} - i - \frac{(m-3)p}{2} + 1 - \left(\frac{mp}{2} - i - \frac{(m-3)p}{2} + 1\right) + \frac{p}{2} - i & \text{for } \frac{p}{2} + 1 \le i \le p \end{cases}$$

$$= \begin{cases} \frac{p}{2} - i + 1 & for\ 1 \le i \le \frac{p}{2} \\ \frac{p}{2} - i & for\ \frac{p}{2} + 1 \le i \le p \end{cases}$$

Hence f* (V) = $\{\pm 1, \pm 2, \ldots, \pm p/2\}$

For $1 \le s,t \le \frac{p}{2}$, when $s \neq t$, clearly $f^*(v_s) \neq f^*(v_t)$ and also

For $\frac{p}{2} + 1 \le s,t \le p$, when $s \neq t$, clearly $f^*(v_s) \neq f^*(v_t)$.

But for $1 \le s \le \frac{p}{2}$ and $\frac{p}{2} + 1 \le t \le p$,

When $s \neq t$, if $f^*(v_s) = f^*(v_t)$ then t = s-1 which is a contradiction to $t > s$.

Therefore $f^*(v_s) \neq f^*(v_t)$.

Hence f* (V) = $\{\pm 1, \pm 2, \ldots, \pm p/2\}$ is a bijection.

Therefore The Cayley digraph Cay (G,S) is super edge graceful when m is odd and p is even.

Case (iv): If m and p are odd

Since m and p are odd, both the number of vertices and the number of edges are also odd. To prove the theorem, we have to prove that there is a bijection f: E $\rightarrow \{0, \pm 1, \pm 2, \ldots, \pm (mp-1)/2\}$ such that the induced function, $f^*(v_i)$ =

$\sum_{j=1}^{m} f(e_{ij})$, $1 \le i \le p$ is a bijection from V to $\{0, \pm 1, \pm 2, \ldots, \pm (p-1)/2\}$ where the summation is taken over the outgoing arcs of v_i.

Define $f : E \to \{0, \pm 1, \pm 2, \ldots, \pm(mp-1)/2\}$ as follows

For $1 \le i \le p$ and $1 \le j \le m-1$

$$f(e_{ij}) = \begin{cases} \frac{mp-1}{2} - i - \frac{p(j-1)}{2} + 1 & \text{for } j = 1,3,\ldots,m-2 \\ -\left(\frac{mp-1}{2} - i - \frac{p(j-2)}{2} + 1\right) & \text{for } j = 2,4,\ldots,m-1 \end{cases}$$

For $j = m$ and $1 \le i \le p$

$$f(e_{ij}) = \left\{ \frac{p+1}{2} - i \quad for\ 1 \le i \le p \right.$$

Then the induced function,

$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$ for $1 \le i \le p$

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$$\begin{cases} \frac{mp-1}{2} - i + 1 - \left(\frac{mp-1}{2} - i + 1\right) + \frac{mp-1}{2} - i - p \\ +1 - \left(\frac{mp-1}{2} - i - p + 1\right) + \cdots \\ \frac{mp-1}{2} + i - \frac{(m-3)p}{2} + 1 - \\ \left(\frac{mp-1}{2} - i - \frac{(m-3)p}{2} + 1\right) + \frac{p+1}{2} - i \\ \text{for } 1 \le i \le p \end{cases}$$

$= \frac{p+1}{2} - i \quad for\ 1 \le i \le p$

Hence $f^*(V) = \{0, \pm 1, \pm 2, \ldots, \pm (p-1)/2\}$.

For $1 \le s, t \le p$, when $s \ne t$, clearly $f^*(v_s) \ne f^*(v_t)$.

Hence $f^*(V) = \{0, \pm 1, \pm 2, \ldots, \pm (p-1)/2\}$ is a bijection.

Therefore The Cayley digraph Cay (G,S) is super edge graceful when m and p are odd.

Hence The Cayley digraph Cay (G,S) is super edge graceful.

B. Theorem

The Cayley digraph Cay (G,S) is even edge graceful.

Proof:

Consider the Cayley digraph Cay (G,S) with p vertices and m number of generators. Therefore the graph has mp arcs totally and every vertex of the graph contains m outgoing and m incoming arcs. . Denote the vertex set of Cay (G,S) as

$V = \{v_1, v_2, \ldots, v_p\}$ and the edge set of Cay (G,S) as

$E = E_{s_1} \cup E_{s_2} \cup \ldots, \cup E_{s_m}$

$= \{e_{11}, e_{12}, \ldots, e_{1m}, e_{21}\ e_{22}, \ldots, e_{2m}, \ldots, e_{p1}, \ldots, e_{pm}\}$

Where e_{ij} is an outgoing arc from i^{th} vertex generated by the generator s_j and

E_{s_1} = set of all outgoing arcs from v_i, $1 \le i \le p$ generated by s_1

E_{s_2} = set of all outgoing arcs from v_i, $1 \le i \le p$ generated by s_2

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E_{s_m} = set of all outgoing arcs from v_i, $1 \le i \le p$ generated by s_m

To prove the Cayley digraph Cay (G,S) with $|S| = m$ and $|V| = p$ is even edge graceful we have to show that there is there is an injection f from the set of edges to $\{1,2,3,\ldots,2mp\}$ such that the values of the induced mapping f* from the vertex set $\{0,1,2,\ldots,2mp-1\}$ given by $f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$ (mod 2mp) over all outgoing arcs of v_i, $1 \le i \le p$ are even and distinct.

Define $f : E \to \{1,2,3,\ldots,2mp\}$ as follows

$f(e_{ij}) = \{2i + 2(j-1)p$; $1 \le i \le p$ and $1 \le j \le m$

Then for every $v_i \in V$, $1 \le i \le p$

$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij}) = \sum_{j=1}^{m} 2i + 2(j-1)p$

$= 2mi + (m-1)mp$

Therefore $f^*(v_i) \equiv 2mi$ (mod 2mp) when m is odd and

$f^*(v_i) \equiv [2mi - mp + m^2p]$ (mod 2mp) when m is even.

$f^*(v_i)$ is always even and distinct for every i, $1 \le i \le p$.

Hence we proved that the Cayley digraph is even edge graceful.

IV. CONCLUSION AND FUTURE WORK

In this paper we provided two labeling on Cayley digraphs and also presented the conditions to make the Cayley digraphs super edge graceful and even edge graceful respectively. We have also explained the applications of labeled Cayley digraphs in the field of interconnection networks. In future the research can be done to obtain the condition at which irregular and oriented digraphs are super edge graceful and even edge graceful.

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