Signed Total Roman Dominating Functions of Corona Product of a Path with a Complete Graph

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Abstract — *Graph theory is a fascinating subject in mathematics. Its Applications in many fields like Physical Sciences, Engineering communications, coding theory, Linguistics, Logical Algebra and Computer networking. Let G be a simple graph with vertex set V and edge set E and the function* $f:V\to[0,1]$ *is called a dominating function (DF) of G, if for each* $v\in V$ *, the sum of the function values over v and the elements incident to v is greater than or equal to one. It is a minimal dominating function (MDF), if for all* $g>f$ *, g is not DF. In this paper, we study the minimal total dominating functions, minimal total roman dominating functions, minimal signed total roman dominating functions of corona product graph of a path with a complete graph and obtain total domination number* $\gamma_t(G)$ *, total roman domination number* $\gamma_{tR}(G)$ *and signed total roman domination number* $\gamma_{stR}(G)$ *of these graphs.*

Keywords — *Corona Product graph, Signed total roman dominating functions, Signed total roman domination number.*

I. INTRODUCTION

The theory of domination in graphs has a wide range of applications. Among these applications, the most often discussed is a communication network. This network consists of communication links between a fixed set of locations. The problem is to select a smallest set of locations at which the transmitters are placed so that every other location in the network is joined by a direct communication link to the location, which has a transmitter. In other words, the problem is to find a minimum dominating set in the graph corresponding to this network.

Generally Product of graphs occurs in discrete mathematics. Frucht & Harary [7] introduced a new product on two graphs G_1 and G_2, called corona product denoted by $G_1 \square G_2$. The corona product of a path P_n with a complete graph K_m is a graph obtained by taking one copy of n-vertex path P_n and n copies of K_m and then joining the ith vertex of P_n to every vertex of ith copy of K_m and it is denoted by $P_n \square K_m$, where $n>0$ and $m>0$.

Allan, Laskar & Hedetniemi [6], Cockayne, Dawes & Hedetniemi [1] and Henning & Kazemi [5] have studied about total domination in graphs. Total roman domination which is suggested by the article in "Total Roman domination in graphs" by Ahangar, Henning, Samodivkin & Yero [2]. Volkmann [3,4] introduced the concept of signed total roman domination in graphs.

A function $f:V\to[0,1]$ is called a total dominating function (TDF) of G, if $f(N(v))=\sum_{u\in N(v)} f(u)\geq 1$, for each $v\in V$. It is a minimal total dominating function (MTDF), if for all $g>f$, g is not TDF. Let $f:V\to\{0,1,2\}$ be a function having the property that for every vertex $v\in V$ with $f(v)=0$, there exists a neighbor $u\in N(v)$ with $f(u)=2$. Such a function is called a total roman dominating function. The weight of a total roman dominating function is the sum $f(V)=\sum_{u\in V} f(u)\geq 1$. The minimum weight of a total roman dominating function on G is called the total roman domination number of G and is denoted $\gamma_{tR}(G)$. Let a function $f:V\to\{-1,1,2\}$ is called a signed total roman dominating function (STRDF) of G , if $f(N(v))=\sum_{u\in N(v)} f(u)\geq 1$, for each $v\in V$ and satisfying the condition that every vertex u for which $f(u)=-1$ is adjacent to at least one vertex v for which $f(v)=2$. A signed total roman dominating function f of G is called a minimal STRDF, if for all $g<f$, g is not a STRDF. The weight of f , denoted $f(G)$, is the sum of the function

value of all vertices in G, that is $f(G)=\sum_{x\in V} f(x)$.

The signed total roman domination number of G, $\gamma_{stR}(G)$, is the minimum weight of a signed total roman dominating function on G.

II. Results

Theorem 1: The total domination number of a graph $G = P_n \square K_m$ is n.

Proof: Consider $G = P_n \square K_m$. Let T denote a total dominating set $G = P_n \square K_m$. Suppose T contains the set of vertices of P_n. By the definition of the graph $G = P_n \square K_m$, every vertex in P_n is adjacent to all vertices of each copy of K_m. That is, the vertices in P_n totally dominates the vertices in all copies of K_m respectively. Thus T becomes a total dominating set of $G = P_n \square K_m$. Also T is a minimal total dominating set of $G = P_n \square K_m$. (By the definition of minimal total dominating set).

Therefore $\gamma_t(G)=n, if\ G=P_n \square K_m$.

Theorem 2: Let T be a minimal total dominating set of $G = P_n \square K_m$ whose vertex set is V and the function $f:V\to[0,1]$ is defined by $f(v)=\begin{cases}1, & \text{if } v\in T,\\ 0, & \text{otherwise.}\end{cases}$ becomes a minimal total dominating function of G and total domination number is $\gamma_t(G)=n$.

Proof: Consider $G = P_n \square K_m$. Let T be a minimal total dominating set of $G = P_n \square K_m$. Clearly this set contains all vertices of P_n and this set is also minimal. Now we consider the vertices according to its degree. Then the following cases are formed.

Case 1: Let $v\in P_n$ be such that $d(v)=m+2$ in G then

$$\sum_{u\in N(v)} f(u)=1+1+\underbrace{0+---+0}_{(m)-times}=2.$$

Case 2: Let $v\in P_n$ be such that $d(v)=m+1$ in G then

$$\sum_{u\in N(v)} f(u)=1+\underbrace{0+---+0}_{(m)-times}=1.$$

Case 3: Let $v\in K_m$ be such that $d(v)=m$ in G then

$$\sum_{u\in N(v)} f(u)=1+\underbrace{0+---+0}_{(m-1)times}=1.$$

Hence for all the above possibilities, we get $\sum_{u\in N(v)} f(u)\ge 1,\ \forall v\in V$.

This implies that the function f is a total dominating function. Now we check for minimality of f, define $g:V\to[0,1]$ by

$$g(v)=\begin{cases}s, & \text{if } v=v_k\in T,\\ 1, & \text{if } v\in T-\{v_k\},\\ 0, & \text{otherwise.}\end{cases}$$

Where $0<s<1$. Since, strict inequality holds at the vertex $v_k\in T$, it follows that $g<f$.

Case 1: Let $v\in P_n$ be such that $d(v)=m+2$ in G.

(i) If

$$v_k\in N(v),\ \text{then}\ \sum_{u\in N(v)} g(u)=s+1+\underbrace{0+---+0}_{(m)-times}=s+1>1.$$

(ii) If

$$v_k\notin N(v),\ \text{then}\ \sum_{u\in N(v)} g(u)=1+1+\underbrace{0+---+0}_{(m)-times}=2.$$

Case 2: Let $v\in P_n$ be such that $d(v)=m+1$ in G.

(i) If $v_k\in N(v)\Rightarrow \sum_{u\in N(v)} g(u)=s+\underbrace{0+---+0}_{(m)-times}=s<1.$

(ii) If $v_k\notin N(v)\Rightarrow \sum_{u\in N(v)} g(u)=1+\underbrace{0+---+0}_{(m)-times}=1.$

Case 3: Let $v\in K_m$ be such that $d(v)=m$ in G.

(i) Let $v_k\in N(v)\Rightarrow \sum_{u\in N(v)} g(u)=s+\underbrace{0+---+0}_{(m-1)-times}=s<1.$

(ii) Let $v_k\notin N(v)\Rightarrow \sum_{u\in N(v)} g(u)=1+\underbrace{0+---+0}_{(m-1)-times}=1.$

This implies that g is not a total dominating function because $\sum_{u\in N(v)} g(u)<1$, for some $v\in V$.

Hence f is a minimal total dominating function on $G = P_n \square K_m$.

Now $\sum_{u\in V(G)} f(u)=\underbrace{1+---+1}_{n-times}+\underbrace{0+---+0}_{n-times\,of\,(m)-zeros}=n.$

Thus n is the minimum value of $\sum_{u\in V(G)} f(u)$;

Finally $\gamma_t(G)=n$.

Theorem 3: Let the function $f:V\to[0,1]$ is defined by $f(v)=\frac{1}{q},\ \forall v\in V$ becomes a total dominating function of $G = P_n \square K_m$, if $q\in(0,m]$ and it is minimal total dominating function if $q=m$. Then the total domination number is $\gamma(G)=\frac{1}{q}n(m+1)$.

Proof: Let f be a function defined in the theorem hypothesis. Consider $G = P_n \square K_m$. Now we consider the vertices according to its degree. Then the following cases are formed.

Case-I: Suppose $0<q<m$. Here $m\ge 1$ and $q<m$.

Case 1: Let $v\in P_n$ be such that $d(v)=m+2$ in G then

$$\sum_{u\in N(v)} f(u)=\frac{1}{q}+\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]=(m+2).\frac{1}{q}>1.$$

Case 2: Let $v\in P_n$ be such that $d(v)=m+1$ in G then

$$\sum_{u\in N(v)} f(u)=\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]=(m+1).\frac{1}{q}>1.$$

Case 3: Let $v\in K_m$ be such that $d(v)=m$ in G then

$$\sum_{u\in N(v)} f(u)=\frac{1}{q}+\left(\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{(m-1)-times}\right)=\frac{m}{q}>1.$$

Hence for all the above possibilities, we get $\sum_{u\in N(v)} f(u)>1,\ \forall v\in V$.

This implies that the function f is a total dominating function. Now we check for minimality of f, define $g:V\to[0,1]$ by

$$g(v)=\begin{cases} s, & \text{if } v=v_k\in V,\\ \frac{1}{q}, & \text{otherwise.}\end{cases}$$

Where $0<s<\frac{1}{q}$. Since, strict inequality holds at the vertex $v_k\in V$, it follows that $g<f$.

Case 1: Let $v\in P_n$ be such that $d(v)=m+2$ in G.

(i) If

$$v_k\in N(v),\ \text{then}\ \sum_{u\in N(v)} g(u)=s+\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]<\frac{1}{q}+\frac{m+1}{q}=\frac{m+2}{q}>1.$$ (ii)

If

$$v_k\notin N(v),\ \text{then}\ \sum_{u\in N(v)} g(u)=\frac{1}{q}+\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]=\frac{m+2}{q}>1.$$

Case 2: Let $v\in P_n$ be such that $d(v)=m+1$ in G.

(i) If

$$v_k\in N(v),\ \text{then}\ \sum_{u\in N(v)} g(u)=s+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]<\frac{1}{q}+\frac{m}{q}=\frac{m+1}{q}>1.$$

(ii) If

$$v_k\notin N(v),\ \text{then}\ \sum_{u\in N(v)} g(u)=\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]=\frac{m+1}{q}>1.$$ Ca

se 3: Let $v\in K_m$ be such that $d(v)=m$ in G.

(i) Let

$$v_k\in N(v)\Rightarrow \sum_{u\in N(v)} g(u)=s+\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{(m-1)-times}<\frac{1}{q}+\frac{(m-1)}{q}=\frac{m}{q}>1.$$

(ii) Let

$$v_k\notin N(v)\Rightarrow \sum_{u\in N(v)} g(u)=\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{(m-1)-times}\right]=\frac{m}{q}=1.$$

This implies that g is also a total dominating function because $\sum_{u\in N(v)} g(u)\geq 1,\ \forall v\in V$.

Hence f is not a minimal total dominating function on $G=P_n\square K_m$.

Case-II: Suppose $q=m$. Substitute $q=m$ in case-I, then we get the following results.

Case 1: Let $v\in P_n$ be such that $d(v)=m+2$ in G then

$$\sum_{u\in N(v)} f(u)=\frac{1}{q}+\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]=(m+2).\frac{1}{m}>1\cdot$$

Case 2: Let $v\in P_n$ be such that $d(v)=m+1$ in G

then $$\sum_{u\in N(v)} f(u)=\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]=(m+1).\frac{1}{m}>1\cdot$$

Case 3: Let $v\in K_m$ be such that $d(v)=m$ in G then

$$\sum_{u\in N(v)} f(u)=\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{(m-1)-times}\right]=\frac{m}{q}=\frac{m}{m}=1.$$

Hence for all the above possibilities, we get $\sum_{u\in N(v)} f(u)\geq 1,\ \forall v\in V$.

This implies that the function f is a total dominating function. Now we check for minimality of f, define $g:V\to[0,1]$ by

$$g(v)=\begin{cases} s, & \text{if } v=v_k\in V,\\ \frac{1}{q}, & \text{otherwise.}\end{cases}$$

Where $0<s<\frac{1}{q}$. Since, strict inequality holds at the vertex $v_k\in V$, it follows that $g<f$.

Case 1: Let $v\in P_n$ be such that $d(v)=m+2$ in G.

(i) If

$$v_k\in N(v)\Rightarrow \sum_{u\in N(v)} g(u)=s+\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]<\frac{1}{q}+\frac{m+1}{q}=\frac{m+2}{m}>1.$$

(ii) If

$$v_k\notin N(v)\Rightarrow \sum_{u\in N(v)} g(u)=\frac{1}{q}+\frac{1}{q}+\left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right]=\frac{m+2}{q}=\frac{m+2}{m}>1.$$

Case 2: Let $v\in P_n$ be such that $d(v)=m+1$ in G.

(i) If

$$v_k \in N(v) \Rightarrow \sum_{u\in N(v)} g(u) = s + \left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right] < \frac{1}{q}+\frac{m}{q}=\frac{m+1}{m}>1.$$

(ii) If

$$v_k \notin N(v) \Rightarrow \sum_{u\in N(v)} g(u) = \frac{1}{q} + \left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{m-times}\right] = \frac{m+1}{q}=\frac{m+1}{m}>1.$$

Case 3: Let $v \in K_m$ be such that $d(v) = m$ in G.

(i) Let

$$v_k \in N(v) \Rightarrow \sum_{u\in N(v)} g(u) = s + \underbrace{\frac{1}{q}+---+\frac{1}{q}}_{(m-1)-times} < \frac{1}{q}+\frac{(m-1)}{q} < \frac{m}{q} < \frac{m}{m} < 1.$$

(ii) Let

$$v_k \notin N(v) \Rightarrow \sum_{u\in N(v)} g(u) = \frac{1}{q} + \left[\underbrace{\frac{1}{q}+---+\frac{1}{q}}_{(m-1)-times}\right] = \frac{m}{q} = \frac{m}{m} = 1.$$

This implies that g is not a total dominating function because $\sum_{u\in N(v)} g(u) < 1$, for some $v \in V$.

Hence f is a minimal total dominating function on G.

Thus $\frac{1}{q}n(m+1)$ is the minimum value of $\sum_{u\in V(G)} f(u)$; Finally $\gamma(G) = \frac{1}{q}n(m+1)$.

Theorem 4: A function $f : V \to \{0,1,2\}$ is defined by

$$f(v) = \begin{cases} 2, & \text{if } v \in P_n \text{ in } G, \\ 0, & \text{otherwise.} \end{cases}$$

is a minimal total roman dominating function of a graph $G = P_n \square K_m$ and total roman domination number is $\gamma_{tR}(G) = 2n$, if m is either even or odd.

Proof: Consider the graph $G = P_n \square K_m$ with $|V|$ number of vertices and $|E|$ number of edges. Let the function f defined in the hypothesis.

Case 1: Let $v \in P_n$ be such that $d(v) = m+2$ in G then

$$\sum_{u\in N(v)} f(u) = 2+2+\left[\underbrace{0+---+0}_{m-times}\right] = 4.$$

Case 2: Let $v \in P_n$ be such that $d(v) = m+1$ in G then

$$\sum_{u\in N(v)} f(u) = 2+\left[\underbrace{0+---+0}_{m-times}\right] = 2.$$

Case 3: Let $v \in K_m$ be such that $d(v) = m$ in G then

$$\sum_{u\in N(v)} f(u) = 2+\left[\underbrace{0+---+0}_{(m-1)-times}\right] = 2.$$

Hence for all the above possibilities, we get $\sum_{u\in N(v)} f(u) > 1,\ \forall v \in V$.

Let u be any vertex in G such that $f(u) = 0$. Then $u \in K_m$ such that $d(u) = m$. Let $v \neq u$ be a vertex in G such that $f(v) = 2$. Then $v \in P_n$ such that $d(v) = (m+1)$ or $(m+2)$, where v any vertex in P_n. We now show that u is adjacent to v. If $v \in P_n$, then v adjacent to u. Since every vertex in P_n is adjacent to every vertex in the corresponding copy of K_m. This implies that the function f is a total roman dominating function. Now we check for minimality of f, define

$$g : V \to \{0,1,2\} \text{ by } g(v) = \begin{cases} 1, & \text{if } v = v_k \text{ of } P_n \text{ in } G, \\ 2, & \text{if } v \in P_n - \{v_k\} \text{ in } G, \\ 0, & \text{otherwise.} \end{cases}$$

Case 1: Let $v \in P_n$ be such that $d(v) = m+2$ in G.

$$\text{If } v_k \in N(v) \Rightarrow \sum_{u\in N(v)} g(u) = 2+1+\left(\underbrace{0+0+---+0}_{m-times}\right) = 3.$$

$$\text{If } v_k \notin N(v) \Rightarrow \sum_{u\in N(v)} g(u) = 2+2+\left(\underbrace{0+0+---+0}_{m-times}\right) = 4.$$

Case 2: Let $v \in P_n$ be such that $d(v) = m+1$ in G.

$$\text{If } v_k \in N(v) \Rightarrow \sum_{u\in N(v)} g(u) = 1+\left(\underbrace{0+0+---+0}_{m-times}\right) = 1.$$

$$\text{If } v_k \notin N(v) \Rightarrow \sum_{u\in N(v)} g(u) = 2+\left(\underbrace{0+0+---+0}_{m-times}\right) = 2.$$

Case 3: Let $v \in K_m$ be such that $d(v) = m$ in G.

$$\text{If } v_k \in N(v) \Rightarrow \sum_{u\in N(v)} g(u) = 1+\left(\underbrace{0+0+---+0}_{m-times}\right) = 1.$$

$$\text{If } v_k \notin N(v) \Rightarrow \sum_{u\in N(v)} g(u) = 2+\left(\underbrace{0+0+---+0}_{m-times}\right) = 2.$$

This implies that $\sum_{u\in N(v)} g(u) \geq 1,\ \forall\, v \in V$. That means g is a total dominating function. But g is not a total roman dominating function, since the total roman dominating function definition fails in the v_k of P_n in G. Because the vertex u in the kth copy of K_m in G for which $f(u) = 0$ is adjacent to a vertex v_k for which $f(v_k) = 1$. Hence f is a minimal total roman dominating function on G.

$$\text{Now } \sum_{u\in V(G)} f(u) = \underbrace{2+---+2}_{n-times} + \left(\underbrace{0+---+0}_{m-times}\right) = 2n.$$

Finally total roman domination number is $\gamma_{tR}(G)=2n$, if m is even or odd.

Theorem 5: A function $f:V\to\{-1,1,2\}$ is defined by

$$f(v_i)=\begin{cases}2, & \text{if } \forall v_i\in P_n,\\ -1, & \text{if } 1\le i\le\left(\frac{m-1}{2}\right) \text{ of each copy of } K_m \text{ in } G,\\ +1, & \text{otherwise.}\end{cases}$$

is a minimal signed total roman dominating function of a graph $G=P_n\square K_m$ and signed total roman domination number is $\gamma_{stR}(G)=3n$, if m is odd.

Proof: Consider the graph $G=P_n\square K_m$ with $|V|$ number of vertices and $|E|$ number of edges. Let f be a function defined in the hypothesis.

Case 1: Let $v_i\in P_n$ be such that $d(v_i)=m+2$ in G then

$$\sum_{u\in N(v_i)} f(u)=2+2+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(+1)\right]=5.$$

Case 2: Let $v_i\in P_n$ be such that $d(v_i)=m+1$ in G then

$$\sum_{u\in N(v_i)} f(u)=2+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(+1)\right]=3.$$

Case 3: Let $v_i\in K_m$ be such that $d(v_i)=m$ in G and $f(v_i)=-1$ or $+1$.

If

$$f(v_i)=-1\Rightarrow\sum_{u\in N(v_i)} f(u)=2+\left[\left(\frac{m-3}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(+1)\right]=4.$$

If

$$f(v_i)=+1\Rightarrow\sum_{u\in N(v_i)} f(u)=2+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m-1}{2}\right)(+1)\right]=2.$$

Hence for all the above possibilities, we get $\sum_{u\in N(v_i)} f(u)\ge 1,\ \forall v_i\in V$.

This implies that the function f is a signed total roman dominating function. Now we check for minimality of f, define $g:V\to\{-1,1,2\}$ by

$$g(v_i)=\begin{cases}2, & \text{if } \forall v_i\in P_n,\\ -1, & \text{if } 1\le i\le\left(\frac{m+1}{2}\right) \text{ of each copy of } K_m \text{ in } G,\\ +1, & \text{otherwise.}\end{cases}$$

Case 1: Let $v_i\in P_n$ be such that $d(v_i)=m+2$ in G then

$$\sum_{u\in N(v_i)} g(u)=2+2+\left[\left(\frac{m+1}{2}\right)(-1)+\left(\frac{m-1}{2}\right)(+1)\right]=3.$$

Case 2: Let $v_i\in P_n$ be such that $d(v_i)=m+1$ in G then

$$\sum_{u\in N(v_i)} g(u)=2+\left[\left(\frac{m+1}{2}\right)(-1)+\left(\frac{m-1}{2}\right)(+1)\right]=1.$$

Case 3: Let $v_i\in K_m$ be such that $d(v_i)=m$ in G and $g(v_i)=-1$ or $+1$.

If

$$g(v_i)=-1\Rightarrow\sum_{u\in N(v_i)} g(u)=2+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m-1}{2}\right)(+1)\right]=2.$$

If

$$g(v_i)=+1\Rightarrow\sum_{u\in N(v_i)} g(u)=2+\left[\left(\frac{m+1}{2}\right)(-1)+\left(\frac{m-3}{2}\right)(+1)\right]=0.$$

This implies that g is not a signed total roman dominating function because $\sum_{u\in N[v_i]} g(u)<1$, for some $v_i\in V$.

Hence f is a minimal signed total roman dominating function on G.

Now

$$\sum_{u\in V(G)} f(u)=\underbrace{2+\cdots+2}_{n-times}+\underbrace{\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(+1)}_{n-times}=3n.$$

Finally signed total roman domination number is $\gamma_{stR}(G)=3n$, if m is odd.

Theorem 6: A function $f:V\to\{-1,1,2\}$ is defined by

$$f(v_i)=\begin{cases}2, & \text{if } \forall v_i\in P_n,\\ -1, & \text{if } 1\le i\le\left(\frac{m}{2}\right) \text{ of each copy of } K_m \text{ in } G,\\ +1, & \text{otherwise.}\end{cases}$$

is a minimal signed total roman dominating function of a graph $G=P_n\square K_m$ and signed total roman domination number is $\gamma_{sR}(G)=2n$, if m is even.

Proof: Let f be a function defined in the hypothesis.

Case 1: Let $v_i\in P_n$ be such that $d(v_i)=m+2$ in G then

$$\sum_{u\in N(v_i)} f(u)=2+2+\left[\left(\frac{m}{2}\right)(-1)+\left(\frac{m}{2}\right)(+1)\right]=4.$$

Case 2: Let $v_i\in P_n$ be such that $d(v_i)=m+1$ in G then

$$\sum_{u\in N(v_i)} f(u)=2+\left[\left(\frac{m}{2}\right)(-1)+\left(\frac{m}{2}\right)(+1)\right]=2.$$

Case 3: Let $v_i\in K_m$ be such that $d(v_i)=m$ in G and $f(v_i)=-1$ or $+1$.

If

$$f(v_i)=-1\Rightarrow\sum_{u\in N(v_i)} f(u)=2+\left[\left(\frac{m}{2}-1\right)(-1)+\left(\frac{m}{2}\right)(+1)\right]=3.$$

If

$$f(v_i)=+1\Rightarrow\sum_{u\in N(v_i)} f(u)=2+\left[\left(\frac{m}{2}\right)(-1)+\left(\frac{m}{2}-1\right)(+1)\right]=1.$$

Hence for all the above possibilities, we get $\sum_{u\in N(v_i)} f(u)\ge 1,\ \forall v_i\in V$.

This implies that the function f is a signed total roman dominating function.

Now we check for minimality of f, define $g: V \to \{-1,1,2\}$ by

$$g(v_i) = \begin{cases} 2, & \text{if } \forall v_i \in P_n, \\ -1, & \text{if } 1 \le i \le \left(\frac{m+2}{2}\right) \text{ of each copy of } K_m \text{ in } G, \\ +1, & \text{otherwise.} \end{cases}$$

Case 1: Let $v_i \in P_n$ be such that $d(v_i) = m+2$ in G then

$$\sum_{u \in N(v_i)} g(u) = 2+2+\left[\left(\frac{m+2}{2}\right)(-1)+\left(\frac{m-2}{2}\right)(+1)\right] = 2 \cdot$$

Case 2: Let $v_i \in P_n$ be such that $d(v_i) = m+1$ in G then

$$\sum_{u \in N(v_i)} g(u) = 2+\left[\left(\frac{m+2}{2}\right)(-1)+\left(\frac{m-2}{2}\right)(+1)\right] = 0 \cdot$$

Case 3: Let $v_i \in K_m$ be such that $d(v_i) = m$ in G and $g(v_i) = -1$ or $+1$.

If

$$g(v_i) = -1 \Rightarrow \sum_{u \in N(v_i)} g(u) = 2+\left[\left(\frac{m}{2}\right)(-1)+\left(\frac{m-2}{2}\right)(+1)\right] = 1 \cdot$$

If

$$g(v_i) = +1 \Rightarrow \sum_{u \in N(v_i)} g(u) = 2+\left[\left(\frac{m+2}{2}\right)(-1)+\left(\frac{m}{2}-2\right)(+1)\right] = -1$$

This implies that g is not a signed total roman dominating function because

$$\sum_{u \in N(v_i)} g(u) < 1, \text{ for some } v_i \in V.$$

Hence f is a minimal signed total roman dominating function on G.

Now

$$\sum_{u \in V(G)} f(u) = \underbrace{2+---+2}_{n-times} + \underbrace{\left(\frac{m}{2}\right)(-1)+\left(\frac{m}{2}\right)(+1)}_{n-times} = 2n.$$

Finally signed total roman domination number is $\gamma_{stR}(G) = 2n$, if m is even.

III. Conclusions

Here we observe that the functions of $G = P_n \square K_m$ defined in certain cases becomes total dominating functions, total roman dominating functions, signed total roman dominating functions. Based on the minimality of these dominating functions we obtained the following results.

(i) $\gamma_{tR}(G) = 2\gamma_t(G)$

(ii) $\gamma_{stR}(G) = \begin{cases} 2\gamma_t(G), & \text{if m is even.} \\ 3\gamma_t(G), & \text{if m is odd.} \end{cases}$

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References

[1] E. J. Cockayne, R. M. Dawes & S. T. Hedetniemi, "Total domination in graphs", Networks, vol. 10, pp. 211-219, Sep. 1980.

[2] H. A. Ahangar, M. A. Henning, V. Samodivkin, & I. G. Yero, "Total Roman domination in graphs", Applicable Analysis and Discrete Mathematics, vol. 10, pp. 501–517, 2016.

[3] L. Volkmann, "Signed total Roman domination in graphs", Journal of Combinatorial Optimization, vol. 32, pp. 855-871, Oct. 2016.

[4] L. Volkmann, "On the signed total Roman domination and domatic numbers of graphs", Discrete Applied Mathematics, vol. 214, pp. 179-186, Dec. 2016.

[5] M. A. Henning & A. P. Kazemi, "k-Tuple total domination in graphs", Discrete Applied Mathematics, vol. 158, pp. 1006-1011, May 2010.

[6] R. B. Allan, R. Laskar, & S. Hedetniemi, "A note on total domination", Discrete Mathematics, vol. 49, pp. 7-13, Mar. 1984.

[7] R. Frucht and F. Harary, "On the corona of Two Graphs", Aequationes Mathematicae, vol. 4, pp. 322 – 325, Oct. 1970.