YATES QUATERNARY COMPLEX HADAMARD MATRIX

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October 24, 2017

Abstract

In this paper we present a construction for a class of $2^k \times 2^k$, $k \geq 2$ quaternary complex Hadamard matrix called Yates quaternary complex Hadamard matrix and also we present its properties. It has been observed that every normalized Yates quaternary complex Hadamard matrix of order 2^k have full row and column sign spectrum. We also show that Yates quaternary complex Hadamard matrix is a jacket matrix.

2010 AMS Subject classification: 15A15, 65F30, 05B20, 15B34
Keywords: Yates Quaternary Complex Hadamard Matrix, Complex Sign Changes, Sign Spectrum, Jacket Matrix.

1 Introduction:

A quaternary complex Hadamard matrix H of order n is a square $n \times n$ matrix with entries from $(\pm 1, \pm i)$ satisfying the orthogonality relation

$$HH^* = H^*H = nI$$

where H^* is the conjugate transpose of H. If H is a quaternary complex Hadamard matrix of order n, then n is either one or even [11]. It has been conjectured that for any even positive integer n there exists a quaternary complex Hadamard matrix of order n [11].
Two quaternary complex Hadamard matrices are said to be H-equivalent if one can be obtained from the other by a finite sequence of following operations.
(1) Permutations of any pair of rows or columns.
(2) Multiplication of any row or column by $(-1 \ or \ i \ or \ -i)$.
A quaternary complex Hadamard matrix is said to be *normalized* if its first row and first column consists of 1 entirely. We can always obtain a normalized quaternary complex Hadamard matrix by multiplying rows or columns by -1 or i or $-i$ if needed. We recall the construction for a class of $2^k \times 2^k$ Hadamard matrices that was originally proposed in [14] and also see [13] for more details and recent applications.
In Statical trials involving several factors it is necessary to study the joint effect of the factors on a response. Usually the factors are represented by two levels , the low level denoted by " $-$ " and the high level denoted by " $+$ ". This design is called 2^2 factorial design. In an experiment, mathematically, the factors are represented as columns of the design matrix and are denoted by uppercase Latin letters. Thus P refers to the effect of a factor P, Q refers to the effect of a factor Q and PQ refers to the PQ interaction. In such an experiment with two factors P, Q, each at two levels, all possible combinations (treatments) of " $+$ " and " $-$ " are represented by lowercase Latin letters. The p corresponds treatment combination of P at the high level and Q at the low level $(+,-)$ and pq corresponds both the factors at the high level $(+,+)$. By convention , (1) denotes both factors at the low level $(-,-)$. This symbol is used through out the 2^k factorial design. Extending these symbols to the case of $k = 3$ factors A, B, C, the standard form of treatment combination is in some order as

$$(1), p, q, pq, r, pr, qr, pqr$$

for $k = 4$ factors P, Q, R, S as

$$(1), p, q, pq, r, pr, qr, pqr, s, ps, qs, pqs, rs, prs, qrs, pqrs$$

and so on.
Now we construct Yates quaternary complex Hadamard matrix.

2 Definitions:

Definition 2.1. ***(Sign Changes of a Complex Sequence:)***
The number of sign changes of a finite complex sequence $\{a_k\}_{k=1}^{n}$, $a_k = \pm 1$ or $\pm i$ is the number obtained by counting the number of times $+1$ or $+i$ is followed by -1 or $-i$ or that of times -1 or $-i$ is followed by $+1$ or $+i$.

Example 1. The following sequences
$1\ 1\ 1\ i\ i\ i$
$1\ 1\ i\ -1\ -i\ -1$
$1\ 1\ -1\ -i\ i\ 1$
$1\ i\ -i\ 1\ -1\ -i\ i\ 1$

have 0, 1, 2 and 4 sign changes, respectively.

Definition 2.2. ***Sign spectrum of a Quaternary Complex Hadamard matrix:***
The row (column) sign spectrum of an $n \times n$ quaternary complex Hadamard matrix H is the sequence of numbers of sign changes appear in its rows (columns).

Definition 2.3. ***Full Sign spectrum of a Quaternary Complex Hadamard matrix:***
An $n \times n$ quaternary complex Hadamard matrix H has full row (column) sign spectrum if its rows (columns) sign spectrum gives all integers $\{0, 1, ..., n-1\}$ in some order.

Example : Consider the following quaternary complex Hadamard matrix H of order 2^3. The last column shows the corresponding row sign changes and the last row shows the corresponding column sign changes of H.

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$$H = \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 7 \\ 1 & -i & -1 & i & 1 & -i & -1 & i & 4 \\ 1 & i & -1 & -i & 1 & i & -1 & -i & 3 \\ 1 & -i & 1 & -i & -1 & i & -1 & i & 6 \\ 1 & i & 1 & i & -1 & -i & -1 & -i & 1 \\ 1 & i & -1 & -i & -1 & -i & 1 & i & 2 \\ 1 & -i & -1 & i & -1 & i & 1 & -i & 5 \\ \hline 0 & 7 & 4 & 3 & 6 & 1 & 2 & 5 & \end{array}\right]$$

It has been observed that the row sign spectrum $S_3 = \{0, 7, 4, 3, 6, 1, 2, 5\}$ of H contains all integers $\{0, 1, 2, 3, 4, 5, 6, 7\}$. So H has a full row sign spectrum. Similarly H has a full column sign spectrum. Thus H has a full row and column sign spectra.

Definition 2.4. ***Jacket Matrix**:[7]*
Let R be a ring with unity 1. A normalized generalized Butson Hadamard matrices $GBH(N, v)$ matrix K indexed by $G = \{1, ..., v\}$ with entries from $N \leq R^*$ is a jacket matrix if it is permutation equivalent to a matrix of the form

$$\tilde{K} = \left[\begin{array}{ccccccc} 1 & 1 & . & . & . & 1 & 1 \\ 1 & * & . & . & . & * & \pm1 \\ & . & & . & & . & \\ & . & & . & & . & \\ & . & & . & & . & \\ 1 & * & . & . & . & * & \pm1 \\ 1 & \pm1 & . & . & . & * & \pm1 \end{array}\right]$$

Where the central entries $*$ are from N. The jacket width of K is $m \geq 1$ if K is permutation equivalent to a jacket matrix $\tilde{K}$ in which rows $1, \ldots, m, v-m+1, \ldots, v$ and columns $1, \ldots, m, v-m+1, \ldots, v$ all consists of ±1 and m is maximal for this property. If K is not permutation equivalent to any jacket matrix, it has jacket width 0.
Since all non-initial ±1 rows and columns of a jacket matrix sum to 0 in R, the order v of a jacket matrix K must be even.

3 Yates Quaternary Complex Hadamard Matrix

In this experiment we use four factors are represented by four different levels, the low level is denoted by $+1$, the high level is denoted by -1, next two levels we introduce and use the levels as, the imaginary high level is denoted by $+i$ and the imaginary low level is denoted by $-i$. Mathematically the factors of an experiment are shown as columns of a matrix and are denoted as upper case letters . Effect of the factors are denoted by upper case letters sa $A, B, C, \ldots$ and the interaction of factors refers by $AB, AC, BC, \ldots$. The treatments are represented by lower letters as , a represents the treatment combination of A at high level denoted by $+1$, B , C , . . . at low level represents by -1. Also b represents the treatment combination of B at imaginary high level denoted by $+i$ and only A at low level denoted by -1 and all others C, D, . . . at imaginary low level and denoted by $-i$. c, d, . . . , corresponds to B is $-i$. The treatment combination ab corresponds to A and B both at high level denoted by $+1, +1$ and C at low level denoted by -1. similarly bc corresponds to B at imaginary high level and C at high level. (1) corresponds to the factor A at low level and B at imaginary low level. The notations are all over same as is used in 2^k factorial design, we extend the notations of 2^k factorial design and use some imaginary notations. Which is in detail below in matrix form.

Yates Complex Construction. The construction is in detailed below for the matrices of order 2^k for $k = 2, 3, 4$. The first column is always denoted by I represents the total average of the entire experiment and denoted by $+1$. The augment of each of the matrix with an additional row contains the number of sign changes corresponding to the column.

For $k = 2$ the Yates quaternary complex matrix of order 2^2 is

$$Y_2 = \left[\begin{array}{c|cccc} & I & A & B & AB \\ \hline (1) & 1 & -1 & -i & i \\ a & 1 & 1 & -1 & -1 \\ b & 1 & -1 & i & -i \\ ab & 1 & 1 & 1 & 1 \\ \hline & 0 & 3 & 1 & 2 \end{array}\right]$$

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For $k = 3$ the Yates quaternary complex matrix of order 2^3 is

$$Y_3 = \left[\begin{array}{c|cccccccc}
 & I & A & B & C & AB & AC & BC & ABC \\ \hline
(1) & 1 & -1 & -i & -1 & i & 1 & i & -i \\
a & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
b & 1 & -1 & i & -1 & -i & 1 & -i & i \\
ab & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ \hline
c & 1 & -1 & -i & 1 & i & -1 & -i & i \\
ac & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
bc & 1 & -1 & i & 1 & -i & -1 & i & -i \\
abc & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
 & 0 & 7 & 3 & 1 & 4 & 6 & 2 & 5
\end{array}\right]$$

For $k = 4$ the Yates quaternary complex matrix of order 2^4, i:e, Y_4 is

$$\left[\begin{array}{c|cccccccccccccccc}
 & I & A & B & C & D & AB & AC & AD & BC & BD & CD & X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline
(1) & 1 & -1 & -i & -1 & -1 & i & 1 & 1 & i & i & 1 & -i & -i & -1 & -i & i \\
a & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
b & 1 & -1 & i & -1 & -1 & -i & 1 & 1 & -i & -i & 1 & i & i & -1 & i & -i \\
ab & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\ \hline
c & 1 & -1 & -i & 1 & -1 & i & -1 & 1 & -i & i & -1 & i & -i & 1 & i & -i \\
ac & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
bc & 1 & -1 & i & 1 & -1 & -i & -1 & 1 & i & -i & -1 & -i & i & 1 & -i & i \\
x_1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ \hline
d & 1 & -1 & -i & -1 & 1 & i & 1 & -1 & i & -i & -1 & -i & i & 1 & i & -i \\
ad & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
bd & 1 & -1 & i & -1 & 1 & -i & 1 & -1 & -i & i & -1 & i & -i & 1 & -i & i \\
x_2 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ \hline
cd & 1 & -1 & -i & 1 & 1 & i & -1 & -1 & -i & -i & 1 & i & i & -1 & -i & i \\
x_3 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
x_4 & 1 & -1 & i & 1 & 1 & -i & -1 & -1 & i & i & 1 & -i & -i & -1 & i & -i \\
x_5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
 & 0 & 15 & 7 & 3 & 1 & 8 & 12 & 14 & 4 & 6 & 2 & 11 & 9 & 13 & 5 & 10
\end{array}\right] \tag{1}$$

Note that in above matrix, we have $x_1 = abc$, $x_2 = abd$, $x_3 = acd$, $x_4 = bcd$, $x_5 = abcd$ and $X_1 = ABC$ $X_2 = ABD$ $X_3 = ACD$ $X_4 = BCD$ $X_5 = ABCD$.
Such type of matrices are called Yates quaternary complex Hadamard matrices. Yates quaternary complex Hadamard matrix denoted by Y_k of order 2^k hold following properties:

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1. The columns of Y_k are pairwise orthogonal.
2. Every column of Y_k (except the first column I) has 2^{k-1} positive entries and 2^{k-1} negative entries.
3. Multiply to any column with I produces the column itself.
4. Multiplying distinct columns provides a column of matrices, that is we say that in Yates quaternary complex Hadamard matrix columns of matrices are closed under multiplication.
5. The matrix Y_k possesses the basis of k column vectors $[A, B, C, D, \ . \ . \ .] = [y_k, y_{k-1}, \ . \ . \ . \ , \ y_1]$. The Yates quaternary complex Hadamard matrix construction develops the remaining $2^k - k - 1$ columns by taking the $\binom{k}{i}$ possible element-wise multiple of the k basis vectors, for $i = 1, 2, \ . \ . \ . \ , \ k$. From the binomial identity $\sum_{i=0}^{k} \binom{k}{i} = 2^k$, we know that there are 2^k column vectors in total, for $i = 0$ we obtain column vector I of all $1'$s an for $i = 1$ we obtain the k column vectors themselves.

Next we explicitly determine the column sign spectrum for Yates quaternary complex Hadamard matrices.

The construction specifies the sign changes of the basis column vectors and of their element wise products.

Lemma.1

(i) The Yates quaternary complex Hadamard matrices of order 2^k for $k \geq 2$ with k basis column vectors $[y_k, y_{k-1}, \ . \ . \ . \ , \ y_1]$ have column sign change spectrum equal to:

$$[2^k - 1, 2^{k-1} - 1, 2^{k-2} - 1, . \ . \ . \ , 1]$$

(ii) The element wise products of the basis columns have sign change spectrum

$$\prod_{m=1}^{l} y_{j_m} = \sum_{i=1}^{l} (-1)^{m-1} 2^{j_m} \ , \ l \ even, \ j_1 > j_2 > \ . \ . \ . \ > j_l$$

$$\prod_{m=1}^{l} y_{j_m} = \sum_{i=1}^{l-1} (-1)^{m-1} 2^{j_m} + 2^l - 1 \ , \ l \ odd, \ j_1 > j_2 > \ . \ . \ . \ > j_l$$

Example 1. Let $[y_4, y_3, y_2, y_1] = [A, B, C, D]$ the basis column vectors of the Yates quaternary complex Hadamard matrix of order 2^4 shown in (1). The below table shows the sign changes of the element wise products of columns.

Columns	y_4y_3	y_4y_2	y_4y_1
Values	$2^4-2^3=8$	$2^4-2^2=12$	$2^4-2=14$
Columns	y_3y_2	y_3y_1	y_2y_1
Values	$2^3-2^2=4$	$2^3-2=6$	$2^2-2=2$
Columns	$y_4y_3y_2$	$y_4y_3y_1$	$y_4y_2y_1$
Values	$2^4-2^3+2^2-1=11$	$2^4-2^3+2-1=9$	$2^4-2^2+2-1=13$
Columns	$y_3y_2y_1$	$y_4y_3y_2y_1$	
Values	$2^3-2^2+2-1=5$	$2^4-2^3+2^2-2=10$	

We now show that the above three matrices formed with the help of Yates quaternary Complex Hadamard matrix are Jacket matrix and also all matrices constructed with the method of Yates quaternary Complex Hadamard matrix are jacket matrix.
Example.1 The matrices Y_2, Y_3 and Y_4 above are jacket matrix.
Proof. It is clear that from the above three examples and by the definition (2.3) that, by using the H-equivalent operations on above three matrices, Y_2, Y_3 and Y_4, produces the Jacket matrix form.

4 conclusion

In this paper we showed that a construction for a class of $2^k \times 2^k$, $k \geq 2$ quaternary complex Hadamard matrix called Yates quaternary complex Hadamard matrix and also we discussed its properties and also showed that every normalized Yates quaternary complex Hadamard matrix of order 2^k have full row and column sign spectrum. lastly we also discussed it as the form of Jacket matrix.

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