The Cube Duplication Solution (A Compass-straightedge (Ruler) Construction)

Kimuya .M. Alex#[1], Josephine Mutembei#[2]
[1] Department of Physical Sciences, Faculty of Physics, Meru University of Science and Technology, Kenya
[2] Department of Mathematics, Meru University of Science and Technology, Kenya

Abstract
This paper objectively presents a provable construction of generating a length of magnitude; $(2)^{1/3} \cong 1.2599 \ldots \cong 1.26$, *as the geometrical solution for the ancient classical problem of doubling the volume of a cube. Cube duplication is believed to be impossible under the stated restrictions of Euclidean geometry, because the Delian constant* $(2)^{1/3}$ *is classified as an irrational number, which was stated to be geometrically irreducible (Pierre Laurent Wantzel, 1837) [1]. Contrary to the impossibility consideration, the solution for this ancient problem is theorem* 4*, in which an elegant approach is presented, as a refute to the cube duplication impossibility statement. Geogebra software as one of the interactive geometry software is used to illustrate the accuracy of the obtained results, at higher accuracies which cannot be perceived using the idealized platonic straightedge and compass construction.*

Key Words: *Doubling a cube, Delian constant, Compass, Straightedge (Ruler), Classical construction, Euclidean number, Irrational number, GeoGebra software*

Abbreviations

$2\,D$	*Two dimensional*
$3\,D$	*Three dimensional*
$3\,s.f$	*Three significant figures*
$5\,s.f$	*Five significant figures*

Notations

$\overline{AB}$ *Used to denotes a straight line (length)*

$\angle ACB$ *Used to denote an angle*

I. Introduction

There are three influential problems in geometry posed by the ancient Greek mathematicians; The problem of doubling a cube, which require the construction of a cube whose volume is twice that of a given cube, the problem of squaring a circle, in which a square of area equal to that of a given circle has to be constructed, and the problem of angle trisection which involves constructing one third of a given angle or the construction of an angle whose size is three times a given angle. However, a high profile contemporaries have closed the door in solving these problems by assuming them impossible for ruler-compass construction [1, 3, and 4]. This paper is focused on the exposition of a geometrical solution for the Delian problem (doubling the volume of a given cube) to a realizable accuracy of $5s.f$, in reference to the numerical value from the scientific calculator, using only a ruler and compass. Through the ages, mathematicians and other practitioners have wrestled the problem of doubling a cube, but no geometrical solution has been found. The deeper need upon taking up this research project was inspired by the objectives; how could the length of a given edge of a cube be geometrically increased to produce a new edge, such that ratio of the new edge to the original edge would be $1.2599 \ldots : 1 \cong 1.26 : 1$. To have the method for solving the Delian problem at this reasonable accuracy, achieved under the specified conditions of Euclidean constructions [5, 10]. The presented method involves the generating a certain constant length from a given face of a cube, which when geometrically added to the original edge would produce the required results. This application is in harmony with the compass equivalence theorem as provided in proposition II in Book I of Euclid's Elements which states "*any construction via 'fixed' compass may be attained with a collapsing compass. That is; it is possible to construct circles of equal radius centered at any point on a plane*". The elementary idea was based on the consideration that; some objects ('atomic' angular units in realm of circles) redistribute in a given plane to produce some significant figures and objects. This is in harmony with the classical geometric division of a given straight segment into a number of even parts. Therefore, some of these units (circles and line segments), if geometrically identified could help solve the Delian problem apparently to a meaningful degree of accuracy. This rise the deeper need for geometrical inspection, for the relationship between curves and lines in a given square plane. Definition 1.1 involves the geometrical definition of a cube, and definition 1.2 brings out the uncertainty in the presented impossibility proof of cube duplication. Theorems 2.0 and 3.0 involves the geometrical construction of the fractions $127/100$ and $5/4$ respectively, as the deductive possibility of constructing the factor $(2)^{\frac{1}{3}}$. The desire to construct these ratios was motivated by the possibility of constructing some algebraically irrational numbers (in algebraic language) such as $\sqrt{2}$ and $\sqrt{5}$ using ruler compass construction. Theorem 4.0 presents an algorithm for solving $(2)^{\frac{1}{3}}$. Geometric transformation relation of enlargement (resizing objects) was used to justify the geometrical accuracy of the generated method, based on results obtained using an interactive computer software. By following the exposed methodology, it was possible to geometrically figure out the geometrical rationality of the Delian constant $(2)^{1/3}$ as $1.2599 \ldots \cong 1.26$ with precision. This work is contained in the formal rules of classical Euclidean geometry.

Definition 1.1: Geometrical Construction of a Cube

Geometrically, a cube is any three-dimensional solid object bounded by six square faces, facets or sides, with three of the edges meeting at each vertex [6]. The cube is the only regular hexahedron and is one of the five Platonic solids. It has 6 faces, 12 edges, and 8 vertices as shown in figure 1:

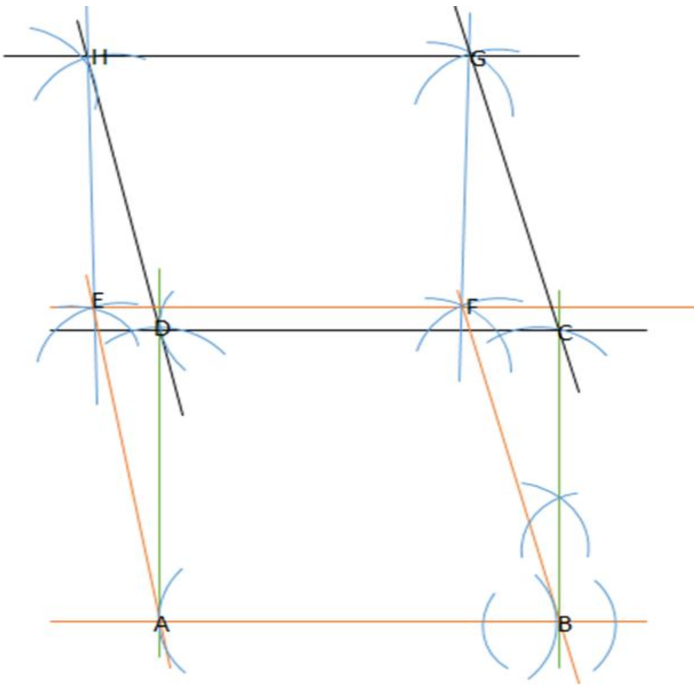

Fig. 1 Geometric View of a Cube

As observed from figure (1), each face of the cube represent a two dimensional plane. Logically, the genetic construction of a cubical structure would involve constructing each of the six facets of the cube, one after another and joining the planes at some common endpoints to produce the eight corners at the vertices.

Definition 1.2 *The Mistake In Pierre Laurent Wantzel's Proof Of Cube Duplication Impossibility Statement (1837)*

This section begins by presenting an interpreted version of the presented cube duplication impossibility statement, to bring out the geometrical misconstruction involved in defining the impossibility statement, based on the algebraic consideration. In his proof, Wantzel used a series of quadratic equations to show that all the geometrically constructible problems are algebraically solvable. He draws the same conclusion for the both problems of cube duplication and the trisection of an angle [7]. It is not geometrically clear the connection between the two problems from his work. Consider the follows interpreted version of the cube duplication impossibility proof.

Theorem 1.0: *Given a cube of volume V it is not geometrically possible to construct a cube of volume $2V$, using only a straightedge (ruler) and a compass.*

Proof: Let the initial cube be of a unit length. Assume that one of the sides of the cube is the line between the coordinates $(0,0)$ and $(1,0)$. The volume of such a cube would be 1, so that constructing a cube of volume 2 would correspond to constructing some point $(\beta, 0)$, such that $\beta^3 = 2$. Let F be the smallest field containing 0 and 1. The minimum polynomial of β over F is $\beta^3 = 2$. This is a degree three polynomial and therefore, based on dimensionality we have:

$$[F(\beta):F] = 3 \qquad (1)$$

Based on application of algebra in plane geometry, the point $(\beta, 0)$ is constructible from the configuration $\{(0,0)\ and\ (1,0)\}$, if $[F(\beta):F]$ is a power of two. However, as seen from equation (1), $[F(\beta):F]$ is a power of three and not two. It can therefore be concluded that the relation $\beta^3 = 2$ is not reducible, and thus the delian constant $\sqrt[3]{2}$ is not geometrically constructible.

From this discussion one can deduce the following points:

The cube duplication impossibility proof does not disproof a geometrical construction aimed at a solution to the cube duplication problem.

The proof involves use of coordinates, which is basically an analytical geometric approach, and not on the classical geometric rigor of analysis.

3. The cube duplication impossibility statement show a degree of misconception in the sense that it does not take into account the difference between the classical geometric problems, and the analytical geometric problems. For instance, the difference between the classical Euclidean geometry and the analytic geometry is that, Euclidean plane geometry is a synthetic geometry in that; it proceeds logically from axioms to propositions without use of the coordinate system, while the analytical geometry completely accomplish the use of coordinate systems. In the ancient Greek's geometry, the only numbers were (positive) integers. Rational number was represented by a ratio of integers. Any other quantity was represented as a geometrical magnitude. The term irrationality did not exist. This point of consideration is presented by (Descartes, 1637). In Book *III* of *La Géométrie* [30], Descartes considers polynomials with integer coefficients. If there is an integer root, that

gives a numerical solution to the problem. But if there are no integral roots, the solutions has be constructed geometrically. A quadratic equation gives rise to a *plane* problem whose solution can be constructed with ruler and compass. Cubic and quartic equations are *solid* problems that require the intersections of conics for their solution. The root of the equation is a certain line segment constructed geometrically, not a number. Up to this point, it is clear that any form of an algebraic proof taming the cube duplication problem as an impossible problem is not a classical geometric solution. It is thus not geometrically valid to assume the cube duplication problem as an impossible problem, since as stated in this context, all geometric problems has to be sought geometrically.

4. The early mathematicians were able to construct algebraic irrationalities of the form: $\sqrt{2}$ and $\sqrt{5}$ (see Annex-1), as employed in the famous Pythagorean theorem, where, the factor $\sqrt{2}$ has no specific value algebraically. This consideration provide clear jurisdiction that, all algebraic irrationalities can be geometrically constructed. It should be noted that, the term exactness in geometry is completely defined by solving a specific problem, based on the set framework. The use of technology is as well limited in the sense that, the CAD methods provide just approximate solutions, and not exact measurements. This paper is in the view that, any classical geometric solution has to be reasonably exact, and not just an eyeballing construction.

In this paper, an attempt is made to show that, the factor $\sqrt[3]{2}$ is geometrically constructible and it is not an irrational number problem. It is the consideration of this paper that, the use of algebra in solving a classical geometric problem is not a geometric verification, but an analysis.

II. Hypothesis

Consider figure (2). The aim of this paper is to provide an algorithm for geometrical solution of the ancient problem of cube duplication, using both the Greek's tools of geometry and computer interactive software (GeoGebra 5.0). From the following figure, the square face$(ABCD)$ represents a surface of the given cube with base $\overline{AB}$. Piont B' is a reflection of point B about point A. Thus $\overline{B'A} = \overline{AB}$. Also, lengths$BL$ and MN are equal to AB. Therefore for a line of magnitude 1.26 to be correctly constructible, $\overline{JA}$ must geometrically equal $\overline{LN}$. Theorem 4 presents a proof to confirm that the generated length $\overline{JB} = \overline{BN} = 1.26$, to an accuracy of $3s.f$.

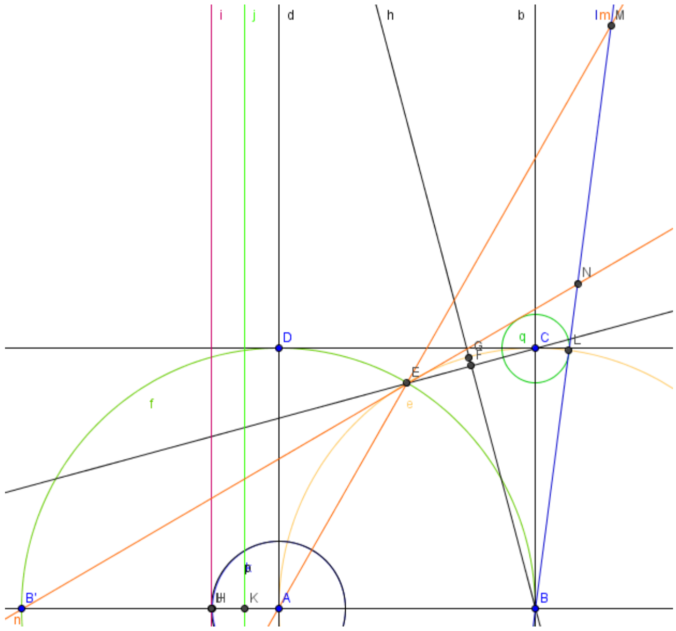

Fig. 2 An illustration of existence of significant points along $\overline{AB}$

III. Materials and Methods

Materials

The required mathematical tools in solving this problem for the proposed methods include;

- Classical compass
- Ruler(straightedge)
- Piece of a drawing paper
- Pencil
- Rubber
- Computer
- GeoGebra Software installed on the PC.

In this paper, though all the provided constructions are typically compass and straightedge methods, the use of GeoGebra software is preferred, purposely for good results visualization.

The GeoGebra Software

GeoGebra is an open source dynamic software for teaching and learning mathematics with suitable features for topics such as geometry, algebra and calculus. It was developed by Markus Hohenwarter and has now been translated into several languages; (Abdul Saha et al. 2010). It is a software designed as a combination of other geometry dynamic software features such as Cabri Geometry, C.a.R, and Geometer's Sketchpad. It is found to be more interactive compared to other geometric software. In this era, geometry is one of the mathematics contents that is concerned with the integration of technology during the teaching and learning process. Therefore, having such good geometry environments (integrated toolboxes), GeoGebra is a suitable dynamic package to be used as an alternative teaching aid that is based on technology. Besides geometrical learning GeoGebra has also other blocksets related to topics of algebra and calculus. As depicted in figure 3, GeoGebra is dynamic geometry software fitted with various characteristics that allow users to construct object such as points, segments, lines, circles, ellipses, angles and other dynamic functions. However, like any other software, GeoGebra has its own limitations in the sense that, it does not give the exact position of a quantity on the graphics pad, but, just approximation. This shows the reason why it is so much difficult to get an exact measurement in classical geometry, as most people are concerned with exactness of quantities, a property that never exist in science. As stated earlier, this tool will specifically be used for results visualization and analysis, and not as part of the provided proofs. Consider figure 3.

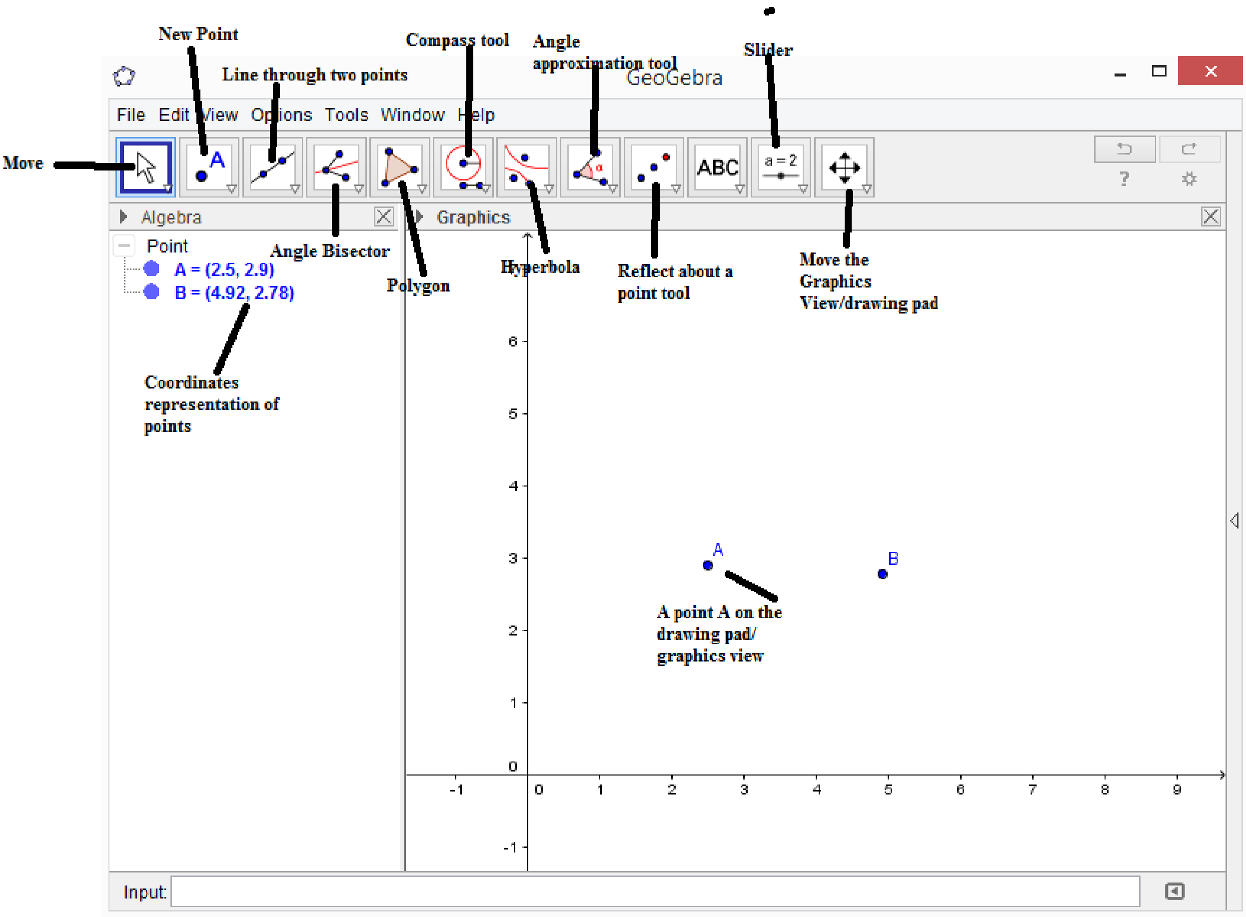

Fig. 3: The GeoGebra Software Interface and Functions

Methodology

Theorem 2.0: Approximate Construction of the fraction $127/100$ form the edge of a given cube

The following method would help the construction of a line of magnitude $\cong 1.27$, given a cube of side one unit.

1. Draw a straight line through two points $(A \text{ and } B)$.
2. Construct square $ABCD$ with $\overline{AB}$ as its base.
3. Locate a point O at the center of the square face by constructing its diagonals.
4. Construct the bisection of line OA at E.
5. Further construct the bisection of EA at F.
6. By placing the pair of compass at C, trace the length CF to cut the edge AB at G.
7. Place the compass at A and make an arc of length GA, to cut line BA externally at H.
8. Join point C to point G using a straight line.

Figure (4) represents the results of this construction.

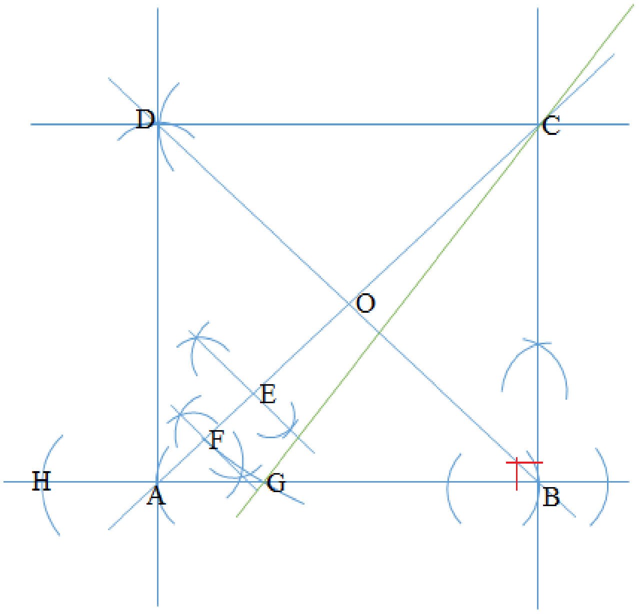

Fig. 4 Construction of the fraction $127/100$

Proof

The purpose of this proof is to show that the fraction $127/100$ is geometrically constructible.

Let the above unit length square $ABCD$ be the face of a cube whose volume is to be doubled.

Let $\overline{AG} = \alpha$, and $\overline{AB} = L$. (2)

The magnitude of diagonal CA can be given by applying the expression $L^2 + L^2 = (\overline{CA})^2$.

Thus $\overline{CA} = L\sqrt{2}$. (3)

The largest fraction of the diagonal $\overline{CA}$ would therefore be

$\overline{CF} = \frac{7}{8} \times L(\sqrt{2})$. (4)

Considering triangle GBC (Right angled at B), the hypotenuse $\overline{CG} = \overline{CF} = \frac{7}{8}L(\sqrt{2})$. (5)

The magnitude of GB can be found using the Pythagoras theorem:

$\overline{GB} = \sqrt{\{(\frac{7}{8}L(\sqrt{2}))^2 - L^2}$. (6)

Therefore, the magnitude of $\overline{AG} = \alpha$ would be given by $\alpha = \overline{L} - \overline{GB}$. (7)

Substituting for $\overline{L} = 1\,units$ we get the value of $\overline{AG} = \alpha = 1 - 0.728869 = 0.2711 \approx 0.27\,units$.

Reflecting point G about point A to produce point H implies adding the length $0.27\,units$ to the edge of the square face. Thus the constructed length is $1.27\,units$

Theorem 3.0: It is possible to construct the fraction $5/4$ from the side of a given cube

The fraction $5/4$ can be geometrically solved by following the presented steps of construction:

1. Draw a straight line through two points(A *and* B), at a distance equal to the side of the given cube.
2. Construct the bisection of line AB at a point C.
3. Further, construct the bisection of AC at point D.
4. Reflect point D about point A to get point D' as shown in diagram 5.

The edge $\overline{D'B} = 1.25\,units$.

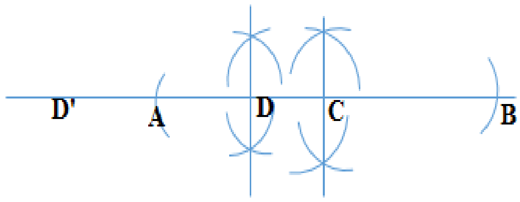

Fig. 5 The construction of the fraction $5/4$

Proof

From the construction results it is evident that, point D is at $(3/4)\overline{AB}$. (8)

This implies that $\overline{AD} = (1/4)\overline{AB}$.

Mapping point D to D' means adding the length AD to line AB. Therefore,

$\overline{D'B} = \overline{AD} + \overline{AB}$. (9)

Taking $\overline{AB} = 1\,unit$, and substituting for $\overline{AB}$ and $\overline{AD}$ in equation (9) we get;

$\overline{D'B} = 1 + (1/4) = 5/4$. (10)

Equation (10) shows that the fraction $5/4$ is geometrically constructible.

Theorem 4.0: It is geometrically possible to construct the ratio $\sqrt[3]{2}:1 = 1.2599\ldots:1 \cong 1.26:1$, for compass and ruler (straightedge) construction

Under this theorem, the objective is to proof that the factor $\sqrt[3]{2}:1 = 1.26\ldots:1$ is constructible using the classical construction tools. Consider the following steps of construction;

1. Given a cube of unit length, construct square face $ABCD$ of sides $1\,unit$.
2. Using the radius AB of the constructed square, place the compass at point A and make an arc through vertices D and B.
3. Without adjusting the compass, position the compass spike at point B and make another arc from point A through C. Label E the point of intersection of the two arc inside the square.
4. Join point E to C using a straight line, and construct its bisection at point F, to cut curve EC at a point G.
5. Using $\overline{EF}$, place the compass at A and mark a point H on the extended edge $\overline{BA}$.
6. Again using chord EG, place the compass at A and mark another arc at a point I on the extended line.

7. Construct the bisection of HI at J as shown in figure (6).
   Line JB is the required length.

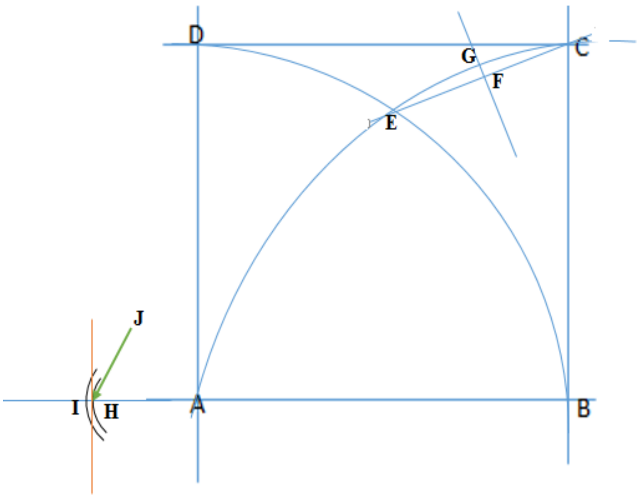

Fig. 6 How to locate point J

Verification of the Method

From figure (6), it is clear that the points I and H are very close that for one to construct the bisection at J, the concept of constructing a perpendicular though a point in a straight line has to be employed. This challenge calls for the use of a geometry software for clarification purpose. GeoGebra software is preferred for use in this paper because of its good geometry tools which makes it more reliable for plane geometry constructions, as stated earlier. The following algorithm was used to generate figure (7).

1. Given a cube of unit length, construct square face $ABCD$ of side $1\ unit$.
2. Using the radius AB of the constructed square, place the compass at point A and make an arc through vertices D and B.
3. Without adjusting the compass, position the compass spike at point B and make another arc from point A through C. Label E the point of intersection of the two arc inside the square.
4. Join point E to C using a straight line, and construct its bisection at point F, to cut curve EC at a point G.
5. Using $\overline{EF}$, place the compass at A and mark a point H on the extended edge $\overline{BA}$.
6. Again using chord EG, place the compass at A and mark another arc at a point I on the extended line.
   7. Construct the bisection of $\overline{HI}$ at J.
   8. Further, construct the bisection of $\overline{JA}$ at K.
   9. Using $\overline{KA}$, place the compass at point C, and make an arc to cut curve AC at a point L outside the square.
10. Draw a straight line from point B thorough point L to meet line AE at point M as shown in figure (7).
11. Reflect point B about point A to get point B'.
12. Draw a straight line from point B' through point E to meet $\overline{BM}$ at point N.
13. Join point B to point E using a straight line.

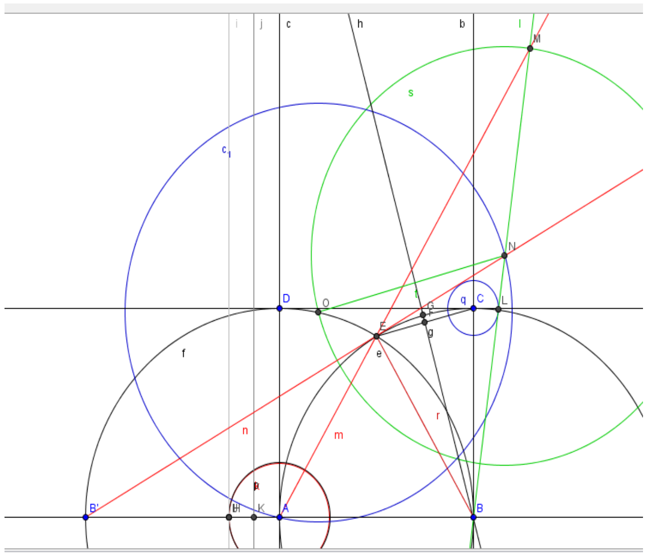

Fig. 7 Illustrative proof for constructing a line of magnitude $1.2599\ldots \cong 1.26$

Proof

From figure (7), several properties are evident: Using the compass equivalence theorem, the purpose of the green and the blue circles with centres N and O respectively is to show that the radii $AB \cong B'A \cong AE \cong BE \cong ON \cong MN$, to accuracy of $3s.f$. The radii MN and ON are inherently generate from the construction. Since the curvature of the blue circle with center O passes through point N (center of the green circle), and the vertex A, it is therefore clear that $\overline{AO} \cong \overline{ON} \cong \overline{NM}$, and $\overline{AO}$ is the initial radius. It can therefore be deduced that, triangle AEB is equilateral (all sides equal), implying, $\angle EAB = \angle ABE = \angle BEA = 60°$. Let $\overline{EM} = y, \overline{NB} = x$ and side of the original cube be n. This proof is to show that; $\overline{NB} = x = n\sqrt[3]{2}$ for $x \neq 0$. Where n is the length of the given cube. Using trigonometry and following the above deductions, the cosine rule can be applied to triangle AMB as follows:

$$\overline{MB}^2 = (\overline{AB}^2 + \overline{AM}^2) - (2 \times \overline{AB} \times \overline{AM} \times cos60°) \quad (11)$$

$$\overline{MB}^2 = (\overline{AB}^2 + \overline{AM}^2) - (2 \times \overline{AB} \times \overline{AM} \times 1/2)$$

$$\overline{MB}^2 - \overline{AB}^2 = \overline{AM}(\overline{AM} - \overline{AB})$$

$$\overline{MB}^2 - \overline{AB}^2 = \overline{AM}.\overline{EM} = \overline{EM}.\overline{AM} \quad (12)$$

Substituting for the lines MB, AM and BA we have:

$\overline{MB} = \overline{MN} + \overline{NB}$, $\overline{AM} = \overline{ME} + \overline{EA}$ and $\overline{BA} = n$.
Further, equation (12) can be rewritten as:

$$(n+x)^2 - n^2 = y(y+n) \qquad (13)$$

$$x^2 + 2nx = y^2 + yn \qquad (14)$$

Applying the Menelaus' Theorem to triangle ABM, with transversal $B'EN$, we have;

$$\overline{AE}/\overline{EM}\,.\,\overline{MN}/\overline{NB}\,.\,\overline{BB'}/\overline{B'A} = 1.$$

Since $\overline{AE} = \overline{BA} = n$, we have
$n/y\,.\,n/x\,.\,2n/n = 1$. This equation reduces to;

$$2n^2 = yx. \qquad (15)$$

Equations (14) and (15) can be solved simultaneously as follows;

$$x^2 + 2nx = (2n^2/x)^2 + (2n^2/x)n$$

$$x^2 + 2nx = 4n^4/x^2 + 2n^3/x$$

$$x^4 + 2nx^3 = 4n^4 + 2n^3x$$

$$x^3(x+2n) = 2n^3(2n+x) \Longrightarrow x^3 = 2n^3$$

The equation reduces to $x = n\sqrt[3]{2}$. Thus the factor $\sqrt[3]{2} \cong 1.26$ is constructed. In this proof, an attempt has been made to correctly map lengths JB and AB onto BN and NM respectively to produce line BM as illustrated in figure (7). This implies $\overline{JA} \equiv \overline{LN}$ to an accuracy of 2 decimal places.

Justification of the proof

This section uses the transformation relation of object similarities and enlargement to justify the correctness of the proposed method for doubling a cube. Consider figure (8) generated after performing the following construction using the GeoGebra software:

1. Given a cube of unit length, construct square face $ABCD$ of sides $1\ unit$.
2. Using the radius AB of the constructed square, place the compass at point A and make an arc through vertices D and B.
3. Without adjusting the compass, position the compass at point B and make another arc from point A through C. Label E the point of intersection of the two arc inside the square.
4. Join point E to C using a straight line, and construct its bisection at point F, to cut curve EC at a point G.
5. Using $\overline{EF}$, place the compass at A and mark a point H on the extended edge $\overline{BA}$.
6. Again using chord EG, place the compass at A and mark another arc at a point I on the extended line.
7. Construct the bisection of HI at J.
8. Use AJ to locate the vertices K and M.

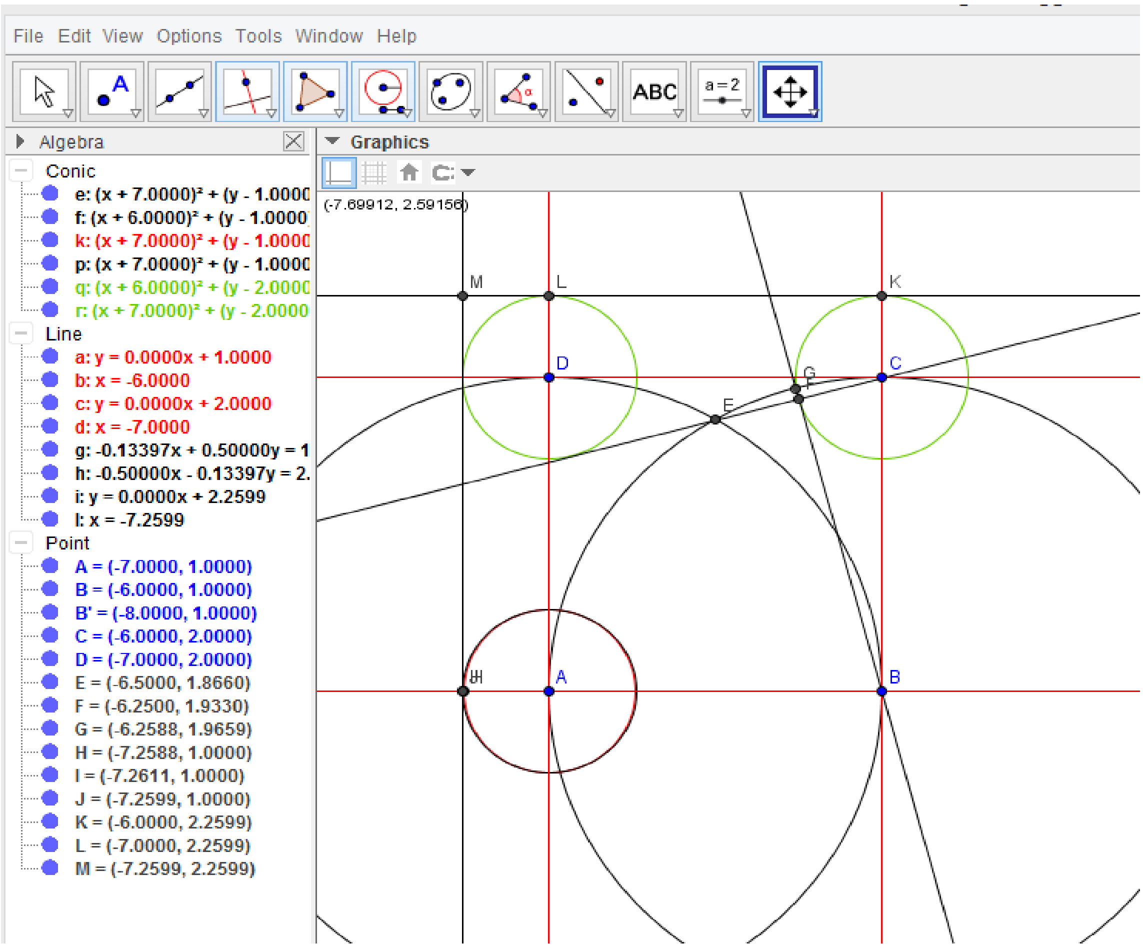

Fig. 8 Analytical Justification of the proposed method

Considering squares $ABCD$ and $JBKM$, the scale factor mapping $ABCD$ onto $JBKM$ can be found. From the GeoGebra window (Algebra view), it is observed, the points D and M have the co-ordinates $(-7.0000, 2.0000)$ and $(-7.2599, 2.2599)$ respectively. Considering these two points, let the scale factor of enlargement be β. Taking the point $B(-6.0000, 1.0000)$ to be the center of enlargement. Therefore, for the $x-axis$:

$$\beta(-7.0000 + 6.0000) = -7.2599 + 6.0000 \quad (16)$$

We get the scale factor as $\beta = 1.2599$.

Therefore the scale factor mapping point D onto M is 1.2599, which is approximately equal to $\sqrt[3]{2} = 1.2599\ ...$. The same value would be obtained if calculated for the $y-axis$.

IV. Results and Discussion

The problem of doubling a cube is a well-known millenary problem about which mathematicians stated as impossible to geometrically resolve because the factor $(2)^{1/3}$ is classified as an irrational number. The incomprehensible proof of impossibility concerned showing that the solution of the constant $(2)^{1/3}$ corresponds to the solution of the cubic equation; $x^3 = 2$, which is not reducible, and thus geometrically unsolvable. Irrational numbers are mathematically defined as being not a finite solution from a division. However, this is not a fashionable definition, as most of number division are an open loop operation that can never be ended [19]. The impossibility proof of doubling a cube was based on three dimensional cubic extensions in abstract algebra, an approach which entirely shifted the problem to solid geometry from its Greek's definition in plane geometry, and therefore the algebraic statement of impossibility has no geometrical validity. Genetically, Euclidean plane geometry obliges the solution of all the compass-ruler constructions to be carried out in a plane (according to the first thee postulates of $book\ I$ of the thirteen books of Euclid elements). This is evident from the fact that no two facets of a cube can share all their four vertices from two different planes. Thus according to this paper, the impossible imprecise classification should not be extended to geometry so that the irrationality definition was stated as "algebraic irrationality is not a constructible number of the geometry". The numerical value of the Delian constant from the numerical calculator settle at $(2)^{1/3} = 1.25992\ ....$ This value lie in the range $(5/4) < (2)^{1/3} < 127/100$, and these two fractions are geometrically constructible as demonstrated earlier. The existence of these two fractions implies the possibility to solve the rationality of the constant $\sqrt[3]{2}$ to a meaningful precision. In respect to the traditional rigor of Greek's geometry, the constant $\sqrt[3]{2}$; (the geometrical magnitude of the Delian constant) has been approximated to the accuracy; $s(2)^{1/3} \cong 1.26$ to $3s.f$, and 1.26 is geometrically achievable following the Euclidian construction rules according to the present proof. This paper presents a geometrically possible method under the set restrictions of Euclidean geometry (in the sense that, all presented constructions have been reduced to the Euclidean postulates of practical geometry), by the construction of the relation $1.2599:1$ as depicted in the justification section.

V. Conclusion

In this paper, an elementary proof for solving the ancient problem of doubling the volume of a given cube, to a certain correctness, is presented. The obtained results indicated that, algebraic irrationalities should not be extended to plane geometric constructions, since subject to application, the desired degree of precision could be possible for compass-straightedge construction. Through the presented discussion, it can be concluded that the Wantzel's statement of impossibility is not geometrically valid, since it does not give the geometrical relationship between the quadratic and the cubic extensions employed in the proof of cube duplication impossibility, with respect to the formal framework of classical geometric constructions. The impossibility proof simply justify a statement and not a concept. The focus of theorem 4 was to convert the problem from the complex $3D$ consideration as presented in the impossibility proof, into a simpler $2D$ problem, and its solution found following the formal Greek's rules of geometry. For centuries, the problem of doubling a cube has been a subject to pseudo mathematical approaches, which do not reach the set limits of accuracy [25]-[29]. The present method has presented a justified geometrical rationality (since all the presented constructions have been reduced to the Euclidean postulates of practical geometry) of the constant $(2)^{1/3}$ as $1.2599\ ...$ to an accuracy of 10^{-3} in reference to the numerical value from the calculator. It can therefore be affirmed that, by following the revealed approach, it is geometrically possible to solve the factors: $[n(2)^{1/3} : 1], [n(3)^{1/3} : 1]$ and other ratios in that order with a meaningful precision, where n is the magnitude of the given cube.

Acknowledgements

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VI. References

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Annex-1: Algebraic Proof of the Impossibility to construct $\sqrt{2}$

Consider the following section of proof:
Suppose that $\sqrt{2}$ is rational, then, $\sqrt{2}$ can be written as $\sqrt{2} = a/b$, where both a and b are whole numbers with no common factor. It follows that;
$2 = (a/b)^2$
Therefore, $2(b^2) = (a^2)$
Clearly, following the algebraic fact that;

1. The product of two odd numbers is odd
2. The product between an odd number and an even number is even
3. The product between two even numbers is even

It can be deduced that; if both a and b have no common factor, then the relation $2(b^2) = (a^2)$ pose a contradiction, since the product $2(b^2)$ is even, because of the number 2. This implies that, a itself must be even. Therefore, it is not possible to construct the factor $\sqrt{2}$, which is classified as irrational.

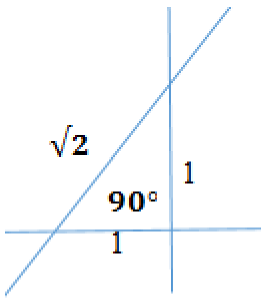

Fig. 9: Geometrical construction of the irrational number $\sqrt{2}$

Annex-1.2: Geometric construct of $\sqrt{2}$
From figure 9, it is evident that, the factor $\sqrt{2}$ is geometrically constructible. The fact that the value of $\sqrt{2}$ cannot be algebraically determined, the geometric interpretation of this case is that, the factor $\sqrt{2}$ is a multiplicative factor, which propagate along any such two dimensional figure of the kind shown in figure 9. Thus in this case, the impossibility proof presented in annex-1 correspond to definition 1.2 in the sense that, it provides proof to a statement, and not a construction.