Strongly g*- Closed Sets in Grill Topological Spaces

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Abstract: *The purpose of this paper is to introduce and study a new class of generalized * - closed sets in a topological space X, defined in terms of a grill G on X. The characterizations of such sets along with certain other properties of them are obtained.*

Keywords: *operator ϕ, topology τ_G, g^* - closed, Gg^* - closed, strongly Gg^* - closed.*

I.INTRODUCTION

It is found from literature that during recent years many topologists are interested in the study of generalized types of closed sets. For instance, a certain form of generalized closed sets was initiated by Levine [3]. Following the trend, we have introduced and investigated a kind of generalized closed set, the definition being formulated in terms of grills. The concept of grill was first introduced by Choquet [1] in the year 1947. M.K.R.S.Veerakumar [6] introduced and investigated between closed sets and g^*- closed sets. The aim of this paper is to introduce and study stronger form of g^*- closed in a topological space X, defined in terms of a grill G on X. From subsequent investigations it is revealed that grills can be used as an extremely useful device for investigation of a number of topological problems.

II.PRELIMINARIES

Definition 2.1: A nonempty collection G of non-empty subsets of a topological space X is called a *Grill* [1] if $(i) A \in G$ and $A \subseteq B \subseteq X \Rightarrow B \in G$ and $(ii) A, B \subseteq X$ and $A \bigcup B \in G \Rightarrow A \in G$ or $B \in G$.

Let G be a grill on a topological space (X,τ). In [5] an operator $\Phi : P(X) \to P(X)$ was defined by $\Phi(A) = \{x \in X / U \bigcap A \in G, \forall U \in \tau(x)\}$, $\tau(x)$ denotes the neighborhood of *x.* Also the map $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \bigcup \Phi(A)$ for all $A \in P(X)$. Corresponding to a grill G, on a topological space (X,τ) there exists a unique topology τ_G on X given by $\tau_G = \{U \subseteq X / \Psi(X \setminus U) = X \setminus U\}$ where for any $A \subseteq X$, $\Psi(A) = A \bigcup \Phi(A) = \tau_G - cl(A)$. Thus a subset A of X is τ_G - closed (resp. τ_G - dense in itself) if $\Psi(A) = A$ or equivalently if $\Phi(A) \subseteq A$ (resp. $A \subseteq \Phi(A)$).

In section III, we introduce and study a new class of generalized*- closed sets, termed $Gg^* -$ closed, in terms of a given grill G, the definition having a close bearing to the above operator Φ. This class of $Gg^* -$closed sets will be seen to properly contain the class of g^* closed sets as introduced in [6]. An explicit form of $Gg^* -$ closed such a closed set is also obtained. In section IV, we introduce and investigate the notion of strongly $g^* -$closed sets in grill topological spaces. The characterizations of such sets along with certain other properties of them are obtained. Also, we investigate its relationship with other closed sets.

Throughout the paper, by a space X we always mean a topological space (X,τ) with no separation properties assumed. If $A \subseteq X$, we shall adopt the usual notations $\text{int}(A)$ and $cl(A)$ respectively for the interior and closure of A in (X,τ). Again $\tau_G - cl(A)$ and $\tau_G - \text{int}(A)$ will respectively denote the closure and interior of A in (X,τ_G). Similarly, whenever we say that a subset A of a space X is

open (or closed), it will mean that A is open (or closed) in (X,τ). For open and closed sets with respective to any other topology on X, eg. τ_G, we shall write τ_G - open and τ_G - closed. The collection of all open neighborhoods of a point x in (X,τ) will be denoted by $\tau(x)$. A subset A of a space (X,τ) is said to be $g-$closed if $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X.
The complement of this set is called $g-open$.

We now append a few definitions and results that will be frequently used in the sequel.

Definition 2.2: A subset A of a space (X,τ,G) is said to be $Gg-closed$ [2] (ie. generalized closed set in grill topological space) if $\Phi(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X.
The complement of this set is called $Gg-open$.

Theorem 2.1: [5] Let (X,τ) be a topological space and G be a grill on X. Then for any $A,B\subseteq X$ the following hold:
(a) $A\subseteq B\Rightarrow\Phi(A)\subseteq\Phi(B)$
(b) $\Phi(A\bigcup B)=\Phi(A)\bigcup\Phi(B)$
(c) $\Phi(\Phi(A)\subseteq\Phi(A)=cl(\Phi(A))\subseteq cl(A)$

Definition 2.3: A subset A of a topological space X is said to be $\theta-closed$ [7] if $A=\theta cl(A)$ where $\theta cl(A)$ is defined as $\theta cl(A)=\{x\in X/cl(U)\bigcap A\neq\phi,\forall U\in\tau\,\&\,x\in U\}$.

Definition 2.4: A subset A of a topological space X is said to be $\theta-open$ [7] if $X\setminus A$ is $\theta-closed$.

Definition 2.5: A subset A of a topological space X is said to be $\delta-closed$ [7] if $A=\delta cl(A)$ where $\delta cl(A)$ is defined as
$\delta cl(A)=\{x\in X/\operatorname{int}cl(U)\bigcap A\neq\phi,\forall U\in\tau\,\&\,x\in U\}$

Definition 2.6: A subset A of a topological space X is said to be $\delta-open$ [7] if $X\setminus A$ is $\delta-closed$.

Lemma 2.1: Let (X,τ) be a space and G be a grill on X. If $A\subseteq X$ is τ_G - dense in itself, then $\Phi(A)=cl\Phi(A)=\tau_G-cl(A)=cl(A)$.

Lemma 2.2: Let (X,τ) be a space and G be a grill on X. (a) Every open set is $g-$open. [4]
(b) Every $g-$open set is $Gg-$open.

III. $g*-$CLOSED SETS WITH RESPECT TO A GRILL

Definition 3.1: Let (X,τ) be a topological space. A subset A of (X,τ) is called *generalized *- closed* ($g*-$ closed) set [6] if $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $g-$ open in X.

Definition 3.2: Let (X,τ) be a topological space and G be a grill on X. Then the subset A of (X,τ) is said to be *generalized *- closed sets with respect to a grill* ($Gg*-$closed) if $\Phi(A)\subseteq U$ whenever $A\subseteq U$ and U is $g-$open in X. A subset A of X is $Gg*-$open if $X\setminus A$ is $Gg*-$ closed.

Theorem 3.1: Let (X,τ) be a topological space and G be a grill on X. Then any non-member of G is $Gg*-$closed.

Proof: Let $A\notin G$ and U be $g-$ open set such that $A\subseteq U$. Then $\Phi(A)\subseteq\phi\subseteq U$. Therefore A is $Gg*-$ closed.

Remark 3.1: The converse of the above theorem need not be true as shown by the following example.

Example 3.1: Let $X=\{a,b,c,d\}$ with topology $\tau=\{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$ and $G=\{\{a\},\{c\},\{a,c\},\{a,b\},\{b,c\},X\}$. In this grill topological space, the subset $\{d\}$ is $Gg*-$ closed but not a member of G.

Theorem 3.2: Let (X,τ) be a topological space and G be a grill on X. Then the following hold.

(a) Every closed set is $Gg*-$closed.

(b) Every $g-$closed set is $Gg*-$closed.

(c) Every $Gg-$ closed set is $Gg*-$ closed.

Proof: (a) Suppose A is a closed set and U is any $g-$ open set. Now, $\Phi(A)\subseteq cl(A)=A\subseteq U$. Therefore, A is $Gg*-$closed.

(b) Suppose A is a $g-$closed set and U is any $g-$ open set. Now, $\Phi(A)\subseteq cl(A)\subseteq U$. Therefore, A is $Gg*-$closed.

(c) Suppose A is a $Gg-$ closed set and U is any $g-$ open set. Now, $\Phi(A)\subseteq U$. Therefore, A is $Gg*-$closed.

Remark 3.2: The converse of the above theorem need not be true as shown by the following example.

Example 3.2: Let $X=\{a,b,c,d\}$ with topology $\tau=\{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$ and $G=\{\{a\},\{c\},\{a,c\},\{a,b\},\{b,c\},X\}$. Then (X,τ) is a topological space and G is grill on X. Then it is easy to verify that

(a) $\{a,d\}$ is $Gg*-$closed but not closed

(b) $\{b\}$ is $Gg*-$closed but not $g-$closed

Theorem 3.3: If A and B are $Gg*-$closed sets in (X,τ,G), then their union $A\bigcup B$ is also $Gg*-$ closed.

Proof: Suppose U is a $g-$ open set containing $A\bigcup B$. Now, $A\bigcup B\subseteq U\Rightarrow A\subseteq U\ \&\ B\subseteq U$. As A and B are $Gg*-$ closed sets, $\Phi(A)\subseteq U\ \&\ \Phi(B)\subseteq U$. Then we have, $\Phi(A\bigcup B)\subseteq U$. Therefore $A\bigcup B$ is also $Gg*-$closed.

Remark 3.3: The intersection of two $Gg*-$closed sets need not be $Gg*-$ closed as shown by the following example.

Example 3.3: Let $X=\{a,b,c,d\}$ with topology $\tau=\{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$ and $G=\{\{a\},\{c\},\{a,c\},\{a,b\},\{b,c\},X\}$. Here, $A=\{a,b,c\},B=\{a,c,d\}$ are $Gg*-$ closed sets but their intersection $A\bigcap B=\{a,c\}$ is not $Gg*-$ closed set.

Theorem 3.4: Let (X,τ,G) be a grill topological space. Then every $\theta-$closed set is $Gg*-$closed.

Proof: Suppose A is a $\theta-$closed set and U be any $g-$open set containing A.
Then $\Phi(A)\subseteq cl(A)\subseteq \theta cl(A)=A\subseteq U$.
Therefore A is $Gg*-$closed.

Remark 3.4: The converse of the above theorem need not be true as shown by the following example.

Example 3.4: Let $X=\{a,b,c,d\}$ with topology $\tau=\{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$ and $G=\{\{a\},\{c\},\{a,c\},\{a,b\},\{b,c\},X\}$. In this grill topological space, the subset $\{a\}$ is $Gg*-$ closed but not $\theta-$closed.

Theorem 3.5: Let (X,τ,G) be a grill topological space. Then every $\delta-$closed set is $Gg*-$closed.

Proof: Suppose A is a $\delta-$closed set and U is any $g-$open set containing A.
Then $\Phi(A)\subseteq cl(A)\subseteq \delta cl(A)=A\subseteq U$.
Therefore A is $Gg*-$closed.

Remark 3.5: The converse of the above theorem need not be true as shown by the following example.

Example 3.5: Let $X=\{a,b,c,d\}$ with topology $\tau=\{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$ and $G=\{\{a\},\{c\},\{a,c\},\{a,b\},\{b,c\},X\}$. In this grill topological space, the subset $\{b\}$ is $Gg*-$ closed but not $\delta-$closed.

IV. STRONGLY $g*-$ CLOSED SETS WITH RESPECT TO A GRILL

Definition 4.1: Let (X,τ) be a topological space and G be a grill on X. Then the subset A of (X,τ) is said to be *strongly generalized *- closed sets with respect to a grill* (strongly $Gg*-$ closed) if $\Phi(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is $Gg-$open in X.

A subset A of X is strongly $Gg*-$open if $X \setminus A$ is strongly $Gg*-$ closed.

Proposition 4.1: For a topological space (X,τ) and G be a grill on X, then the following hold.

(a) Every closed set is strongly $Gg*-$closed.

(b) Every $g-$ closed set is strongly $Gg*-$closed.

(c) Every $Gg-$ closed set is strongly $Gg*-$ closed.

(d) Every $Gg*-$ closed set is strongly $Gg*-$ closed.

(e) Every $\theta-$closed set is strongly $Gg*-$ closed.

(f) Every $\delta-$closed set is strongly $Gg*-$ closed.

(g) Any non member of G is strongly $Gg*-$ closed.

Proof: (a) The proof is immediate from the definition of closed set.

(b) Suppose A is $g-$ closed set in X and U is an open set containing A. Then $cl(A) \subseteq U$. Now, $\Phi(\text{int } A) \subseteq \Phi(A) \subseteq cl(A) \subseteq U$. Therefore A is strongly $Gg*-$closed set .

(c) Suppose A is $Gg-$ closed set in X and U is a $g-$open set containing A. Then $\Phi(A) \subseteq U$. Now, $\Phi(\text{int } A) \subseteq \Phi(A) \subseteq U$. Therefore A is strongly $Gg*-$closed set .

(d) Suppose A is $Gg*-$ closed set in X and U is a $g-$open set containing A. Then $\Phi(A) \subseteq U$. Now, $\Phi(\text{int } A) \subseteq \Phi(A) \subseteq U$. Therefore A is strongly $Gg*-$closed set .

(e) Suppose A is $\theta-$ closed set and U is any $Gg-$ open set containing A. Then $\Phi(\text{int } A) \subseteq \Phi(A) \subseteq cl(A) \subseteq \theta cl(A) = A \subseteq U$. Therefore A is strongly $Gg*-$closed.

(f) Suppose A is $\delta-$ closed set and U is any $Gg-$ open set containing A. Then $\Phi(\text{int } A) \subseteq \Phi(A) \subseteq cl(A) \subseteq \delta cl(A) = A \subseteq U$. Therefore A is strongly $Gg*-$closed.

(g) Suppose $A \notin G$ and U is any $g-$open set such that $A \subseteq U$. Then $\Phi(\text{int } A) \subseteq \Phi(A) \subseteq \phi \subseteq U$. Therefore A is strongly $Gg*-$ closed.

Example 4.1: Let $X=\{a,b,c,d\}$ with topology $\tau=\{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$ and $G=\{\{a\},\{c\},\{a,c\},\{a,b\},\{b,c\},X\}$. Then (X,τ) is a topological space and G is grill on X. Then it is easy to verify that

(a) $\{b,c\}$ is not closed but is strongly $Gg*-$ closed.

(b) $\{b\}$ is not $g-$ closed but is strongly $Gg*-$closed.

(c) $\{c\}$ is not $Gg-$ closed but is strongly $Gg*-$closed.

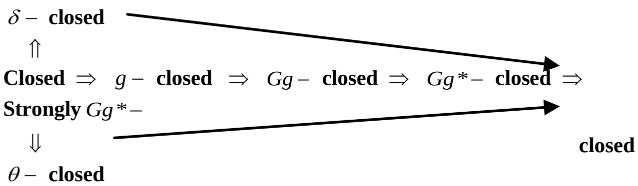
(d) $\{b,c\}$ is not $Gg*-$closed but is strongly $Gg*-$closed.

(e) $\{a,c,d\}$ is not $\theta-$ closed but is strongly $Gg*-$closed.

(f) $\{a,b,c\}$ is not $\delta-$ closed but is strongly $Gg*-$closed.

(g) {b} is not in G but is strongly $Gg*-$ closed.

Remark 4.1: From the discussions and known results we have the following implications.



Theorem 4.1: Let (X,τ,G) be a grill topological space. Then every τ_G closed set is strongly $Gg*-$ closed.

Proof: Suppose A is a τ_G closed set and U is a $Gg-$ open set containing A. Then $\Psi(A)=A \Rightarrow \Phi(A) \subseteq A$. Also $\Phi(\text{int } A) \subseteq \Phi(A) \subseteq A \subseteq U$. Therefore A is strongly $Gg*-$closed.

Theorem 4.2: If a subset A of a grill topological space X is both strongly $Gg*-$ closed and open then it is $Gg*-$closed.

Proof: Suppose A is strongly $Gg*-$ closed and open in X. Let U be an $Gg-$ open set containing A. As A is strongly $Gg*-$ closed, $\Phi(\text{int}(A)) \subseteq U$, $A \subseteq U$ and U is $Gg-$open in X. Now, $\Phi(A) \subseteq U$, (since A is open in X.) U is $Gg-$ open in X. Therefore A is $Gg*-$ closed.

Theorem 4.3: If A and B are strongly $Gg*-$closed set in (X,τ,G), then their union $A \bigcup B$ is also strongly $Gg*-$closed set.

Proof: Suppose A and B are strongly $Gg*-$closed set. Let U be any $Gg-$open set containing $A \bigcup B$. Then $A \bigcup B \subseteq U$. This implies $A \subseteq U$ and $B \subseteq U$. Since A and B are strongly $Gg*-$closed set, $\Phi(\text{int } A) \subseteq U$ and $\Phi(\text{int } B) \subseteq U$, which implies

$\Phi(\text{int } A \bigcup \text{int } B) \subseteq U \Rightarrow \Phi(\text{int}(A \bigcup B)) \subseteq U$.

Therefore $A \bigcup B$ is strongly $Gg*-$closed set.

Remark 4.2: The intersection of two strongly $Gg*-$ closed sets need not be strongly $Gg*-$ closed as shown by the following example.

Example 4.2: Let $X = \{a,b,c,d\}$ with topology $\tau = \{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$ and $G = \{\{a\},\{c\},\{a,c\},\{a,b\},\{b,c\},X\}$. Here, $A = \{a,b,c\}, B = \{a,c,d\}$ are strongly $Gg*-$ closed sets but their intersection $A \bigcap B = \{a,c\}$ is not strongly $Gg*-$closed set.

Theorem 4.4: A subset A is strongly $Gg*-$closed in (X,τ,G) iff $\Phi(\text{int } A) \setminus A$ contains no non-empty closed set.

Proof: Necessary: Suppose that F is non-empty closed set of $\Phi(\text{int } A)$. Now, $F \subseteq \Phi(\text{int } A) \setminus A$ implies $F \subseteq \Phi(\text{int } A) \bigcap (X \setminus A)$. Thus $F \subseteq \Phi(\text{int } A)$. Now, $F \subseteq (X \setminus A) \Rightarrow A \subseteq X \setminus F$. Here $X \setminus F$ is $Gg-$ open and A is strongly $Gg*-$ closed, we have $\Phi(\text{int } A) \subseteq X \setminus F$. Thus $F \subseteq X \setminus \Phi(\text{int } A)$. Hence $F \subseteq X \setminus \Phi(\text{int } A) \bigcap (\Phi(\text{int } A))$. Therefore $F = \phi$. Hence $\Phi(\text{int } A) \setminus A$ contains no non-empty closed set.

Sufficient: Let $A \subseteq U$ and U is $Gg-$ open. Suppose that $\Phi(\text{int } A)$ is not contained in U. Then $X \setminus \Phi(\text{int } A)$ is a non-empty closed set of $\Phi(\text{int } A) \setminus A$, which is a contradiction. Therefore $\Phi(\text{int } A) \subseteq U$ and hence A is strongly $Gg*-$ closed.

Theorem 4.5: Suppose that $B \subseteq A \subseteq X, B$ is strongly $Gg*-$closed set relative to A and that both open and strongly $Gg*-$closed subset of X, then B is strongly $Gg*-$closed set relative to X.

Proof: Suppose $B \subseteq U$ and U is an open set in X. Given that $B \subseteq A \subset X$, therefore $B \subseteq A$ and $B \subseteq U$. This implies $B \subseteq A \bigcap U$. Since B is strongly $Gg*-$closed set relative to A,

$\Phi(\text{int } B) \subseteq A \bigcap U$.

That is $A \bigcap \Phi(\text{int } B) \subseteq A \bigcap U$.

This implies $A \bigcap \Phi(\text{int } B) \subseteq U$. Thus

$(A \bigcap \Phi(\text{int } B)) \bigcup (X \setminus \Phi(\text{int } B)) \subseteq U \bigcup (X \setminus \Phi(\text{int } B))$

which implies

$A \bigcup (X \setminus \Phi(\text{int } B)) \subseteq U \bigcup (X \setminus \Phi(\text{int } B))$. Since A is strongly $Gg*-$closed set in X, we have

$\Phi(\text{int } A) \subset U \bigcup (X \setminus \Phi(\text{int } B))$.

Also, $B \subseteq A \Rightarrow \Phi(\text{int } B) \subseteq \Phi(\text{int } A)$.

Thus

$\Phi(\text{int } B) \subseteq \Phi(\text{int } A) \subseteq U \bigcup (X \setminus \Phi(\text{int } B))$,

which implies $\Phi(\text{int } B) \subseteq U \bigcup (X \setminus \Phi(\text{int } B))$.

Therefore B is strongly $Gg*-$closed set relative to X.

Corollary 4.1: Let A be strongly $Gg*-$closed set and F is closed. Then $A \bigcap F$ is strongly $Gg*-$ closed set.

Proof: To show that $A \bigcap F$ is strongly $Gg*-$ closed set, we have to show that $\Phi(\text{int } A \bigcap F) \subseteq U$, whenever $A \bigcap F \subseteq U$, and U is $Gg-$ open. $A \bigcap F$ is closed in A and so $A \bigcap F$ is strongly $Gg*-$closed in A. By the Theorem 4.5 $A \bigcap F$ is strongly $Gg*-$closed in X.

Theorem 4.6: If A is strongly $Gg*-$closed and $A \subseteq B \subseteq \Phi(\text{int } A)$, then B is strongly $Gg*-$closed.

Proof: Given that $B \subseteq \Phi(\text{int } B)$, then $\Phi(\text{int } B) \subseteq \Phi(\Phi(\text{int } A)) \Rightarrow \Phi(\text{int } B) \subseteq \Phi(\text{int } A)$. Since $A \subseteq B$, we have $\Phi(\text{int } B) \setminus B \subseteq \Phi(\text{int } A) \setminus A$. As A is strongly $Gg*-$closed, then by Theorem 4.4 $\Phi(\text{int } A) \setminus A$ contains no non-empty closed set and also $\Phi(\text{int } B) \setminus B$ contains no non-empty closed set. Again by Theorem 4.4, B is strongly $Gg*-$closed set.

Theorem 4.7: Let $A \subseteq Y \subseteq X$ and suppose that A is strongly $Gg*-$closed set in X then A is strongly $Gg*-$closed set relative to Y.

Proof: Given that $A \subseteq Y \subseteq X$ and suppose that A is strongly $Gg*-$closed set in X. Let $A \subseteq Y \cap U$ where U be $Gg-$open in X. Since A is strongly $Gg*-$closed set in $X, A \subseteq U \Rightarrow \Phi(\text{int } A) \subseteq U$. That is $Y \cap \Phi(\text{int } A) \subseteq Y \cap U$. Therefore A is strongly $Gg*-$closed set relative to Y.

V.CONCLUSIONS

Through the above findings, this paper has attempted to compare strongly $Gg*-$closed set with other closed sets in topological spaces and also in terms of a grill. An attempt of this paper is to state that the definitions and results that shown in this paper, will result in obtaining several characterizations and enable to study various properties as well. In further study the various functions and spaces may be introduced.

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