Certain infinite integral involving the generalized multiple Zeta-function, a general class of polynomials and multivariable Aleph-functions

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ABSTRACT
In the present paper we evaluate an infinite integral involving the product of generalized multivariable Zeta-function, multivariable Aleph-functions and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.

Keywords:Multivariable Aleph-function, general class of polynomial, generalized multiple Zeta-function.

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1.Introduction and preliminaries.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [3] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} \mathrm{y}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r \end{array}\right|$

$$\begin{array}{ll}[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}] & ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]: \\ \cdots\cdots\cdots\cdots\cdots\cdots & ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\alpha_1}, \ldots, |y_r|^{\alpha_r}), max(|y_1|, \ldots, |y_r|) \rightarrow 0$

$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\beta_1}, \ldots, |y_r|^{\beta_r}), min(|y_1|, \ldots, |y_r|) \rightarrow \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1,\cdots,y_r) = \sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\psi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})$$

$$\times\ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})y_1^{-\eta_{G_1,g_1}}\cdots y_r^{-\eta_{G_r,g_r}} \tag{1.6}$$

Where $\psi(.,\cdots,.), \theta_i(.), i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1} = \frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_r,g_r} = \frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i] \neq \delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)

for $j \neq m_i, m_i = 1,\cdots\eta_{G_i,g_i}; p_i, n_i = 0,1,2,\cdots,; y_i \neq 0, i=1,\cdots,r$ (1.8)

Consider the Aleph-function of s variables

$$\aleph(z_1,\cdots,z_s) = \aleph^{0,N:M_1,N_1,\cdots,M_s,N_s}_{P_i,Q_i,\iota_i;r':P_{i^{(1)}},Q_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;P_{i^{(s)}},Q_{i^{(s)}};\iota_{i^{(s)}};r^{(s)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s\end{array}\right|$$

$$\begin{array}{ll}[(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(r')})_{1,N}] & ,[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r')})_{N+1,P_i}]:\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & ,[\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(r')})_{M+1,Q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,N_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,N_s}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_i^{(s)}}]\\ [(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,M_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,M_s}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_i^{(s)}}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_r}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

with $\omega = \sqrt{-1}$

$$\zeta(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{N}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=N+1}^{P_i}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{Q_i}\Gamma(1-v_{ji}+\sum_{k=1}^{s}v_{ji}^{(k)}t_k)]} \tag{1.10}$$

and $$\phi_k(t_k) = \frac{\prod_{j=1}^{M_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{N_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]} \tag{1.11}$$

Suppose , as usual , that the parameters

$u_j, j=1,\cdots,P; v_j, j=1,\cdots,Q;$

$a_j^{(k)}, j = 1, \cdots, N_k; a_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, P_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, Q_{i^{(k)}}; b_j^{(k)}, j = 1, \cdots, M_k;$

with $k = 1 \cdots, s, i = 1, \cdots, r', i^{(k)} = 1, \cdots, r^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} + \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} + \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} - \sum_{j=1}^{M_k} \beta_j^{(k)}$$

$$- \iota_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.12}$$

The reals numbers τ_i are positives for $i = 1, \cdots, r$, $\iota_{i^{(k)}}$ are positives for $i^{(k)} = 1 \cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - u_j + \sum_{i=1}^{s} \mu_j^{(k)} t_k)$with $j = 1$ to N and $\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} t_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

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$$+ \sum_{j=1}^{M_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, s, i = 1, \cdots, r, i^{(k)} = 1, \cdots, r^{(k)} \tag{1.13}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s}), max(|z_1|, \cdots, |z_s|) \rightarrow 0$

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}), min(|z_1|, \cdots, |z_s|) \rightarrow \infty$

where, with $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)} / \beta_j^{(k)})], j = 1, \cdots, M_k$ and

$\beta'_k = max[Re((a_j^{(k)} - 1) / \alpha_j^{(k)})], j = 1, \cdots, N_k$

We will use these following notations in this paper

$$U = P_i, Q_i, \iota_i; r'; V = M_1, N_1; \cdots; M_s, N_s \tag{1.15}$$

$$\mathrm{W} = P_{i^{(1)}}, Q_{i^{(1)}}, \iota_{i^{(1)}}; r^{(1)}, \cdots, P_{i^{(r)}}, Q_{i^{(r)}}, \iota_{i^{(s)}}; r^{(s)} \tag{1.16}$$

$$A = \{(u_j; \mu_j^{(1)}, \cdots, \mu_j^{(s)})_{1,N}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(s)})_{N+1,P_i}\} \tag{1.17}$$

$$B = \{\iota_i(v_{ji}; v_{ji}^{(1)}, \cdots, v_{ji}^{(s)})_{M+1,Q_i}\} \tag{1.18}$$

$$C = (a_j^{(1)}; \alpha_j^{(1)})_{1,N_1}, \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_{i^{(1)}}}, \cdots, (a_j^{(s)}; \alpha_j^{(s)})_{1,N_s}, \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_{i^{(s)}}} \tag{1.19}$$

$$D = (b_j^{(1)}; \beta_j^{(1)})_{1,M_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_{i^{(1)}}}, \cdots, (b_j^{(s)}; \beta_j^{(s)})_{1,M_s}, \iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_{i^{(s)}}} \tag{1.20}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_s) = \aleph_{U:W}^{0,N:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A : C} \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ \mathrm{z}_s & \mathrm{B : D} \end{array}\right) \tag{1.21}$$

The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[y_1, \cdots, y_t] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_t=0}^{[N_t/M_t]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_t)_{M_tK_t}}{K_t!}$$

$$A[N_1, K_1; \cdots; N_t, K_t] y_1^{K_1} \cdots y_t^{K_t} \tag{1.22}$$

Where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_t, K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_t)_{M_tK_t}}{K_t!} A[N_1, K_1; \cdots; N_t, K_t] \tag{1.23}$$

In the document , we note :

$$G(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) \theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) \tag{1.24}$$

where $\phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}), \theta_1(\eta_{G_1,g_1}), \cdots, \theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

2. Generalized multiple Zeta-function

Bin Saad et al [1] have defined the generalized multiple Zeta-function $\phi_{\mu,\lambda_1,\cdots,\lambda_s}^{(s)}(x_1, \cdots, x_s, z, a)$ by

$$\phi_{\mu,\lambda_1,\cdots,\lambda_s}^{(s)}(x_1, \cdots, x_s, z, a) = \sum_{p_1,\cdots,p_s=0}^{\infty} \frac{(\mu)_{p_1+\cdots+p_s} x_1^{p_1} \cdots x_s^{p_s}}{p_1! \cdots p_s! (a + \lambda_1 p_1 + \cdots + \lambda_s p_s)^z} \tag{2.1}$$

where $|x_i| < 1, i = 1 \cdots, s, s \in \mathbb{Z}^+, \mu \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ and

$a \in \mathbb{C} \setminus \{-(\lambda_i, p_i), p_i \in \ N \cup \{0\}\}, Re(z) > 0$

3. Required integral

We have the following integral, see (Marichev et [2] ,2.6.16, Eq.35, page 519)

Lemme

$$\int_0^{\infty} \frac{x}{(x^2+y^2)^{\rho}} ln \left[\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x} \right] \mathrm{d}x = \frac{y^{2-2\rho}}{\rho-1} B(\rho-1,\rho-1) \tag{3.1}$$

with $Re(y) > 0, Re(\rho) > 1$

4. Main integral

Let $X = \dfrac{1}{(x^2+y^2)}$

We have the following formula

Theorem

$$\int_0^{\infty} \frac{x}{(x^2+y^2)^{\rho}} ln \left[\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x} \right] \phi^{(s)}_{\mu,\lambda_1,\cdots,\lambda_s}(x_1 X_1^{\rho_1},\cdots,x_s X_s^{\rho_s},z,a) S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t} \left(\begin{array}{c} \mathrm{y}_1 X^{\gamma_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_t X^{\gamma_t} \end{array} \right)$$

$$\aleph^{0,\mathfrak{n}:v}_{u:w} \left(\begin{array}{c} \mathrm{z}_1 X^{\alpha_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r X^{\alpha_r} \end{array} \right) \aleph^{0,N:V}_{U:W} \left(\begin{array}{c} \mathrm{Z}_1 X^{\eta_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_R X^{\eta_R} \end{array} \right) \mathrm{d}x = y^{2-2\rho} \sum_{p_1,\cdots,p_s=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r}$$

$$\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_t=0}^{[N_t/M_t]} \frac{(\mu)_{p_1+\cdots+p_s}}{p_1!\cdots p_s!(a+\lambda_1 p_1+\cdots+\lambda_s p_s)^z} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} G(\eta_{G_1,g_1},\cdots \eta_{G_r,g_r}) a_1$$

$$x_1^{p_1} \cdots x_s^{p_s}\, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}} y_1^{K_1} \cdots y_t^{K_t}\, y^{-2(\rho+\sum_{i=1}^{s}\rho_i p_i+\sum_{i=1}^{t} K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i)} \aleph^{0,N+2:V}_{U_{21}:W} \left(\begin{array}{c} \frac{Z_1}{y^{2\eta_1}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \frac{Z_R}{y^{2\eta_R}} \end{array} \right|$$

$$(2\text{-}\rho - \textstyle\sum_{i=1}^{s}\rho_i p_i - \sum_{i=1}^{t} K_i\gamma_i - \sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i;\eta_1,\cdots,\eta_R),$$

$$\cdot\ \cdot\ \cdot$$

$$\cdot\ \cdot\ \cdot$$

$$(3\text{-}2(\ \rho + \textstyle\sum_{i=1}^{s}\rho_i p_i + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i);2\eta_1,\cdots,2\eta_R)$$

$$\left. \begin{array}{c} (3\text{-}\rho - \sum_{i=1}^{s}\rho_i p_i - \sum_{i=1}^{t} K_i\gamma_i - \sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i;\eta_1,\cdots,\eta_R), A : C \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{B} : \mathrm{D} \end{array} \right) \tag{4.1}$$

where $U_{21} = P_i + 2; Q_i + 1; \iota_i; r'$

Provided that

a) $min\{\rho_i\gamma_j, \alpha_k, \eta_l\} > 0, i = 1, \cdots, s, j = 1, \cdots, t, k = 1, \cdots, r, l = 1, \cdots, R\,, Re(y) > 0, Re(z) > 0$

b)$Re\left(\rho + \sum_{i=1}^{s} \rho_i p_i\right) + \sum_{i=1}^{r} \alpha_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) + \sum_{i=1}^{R} \eta_i \min_{1 \leqslant j \leqslant M_i} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > 1$

c) $|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i = 1, \cdots, r$

d) $|argZ_k| < \frac{1}{2} B_i^{(k)} \pi$, where $B_i^{(k)}$ is defined by (1.13) ; $i = 1, \cdots, R$

e) The multiple serie occuring on the right-hand side of (3.1) is absolutely and uniformly convergent.

f) $|x_i| < 1, i = 1 \cdots, s, s \in \mathbb{Z}^+, \mu \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ and $a \in \mathbb{C} \setminus \{-(\lambda_i, p_i), p_i \in \ N \cup \{0\}\}$

Proof

First, expressing the generalized multiple Zeta-function in multiple serie with the help of equation (2.1), the Aleph-function of r variables in series with the help of equation (1.6), the general class of polynomial of several variables $S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}$ with the help of equation (1.22) and the Aleph-function of s variables in Mellin-Barnes contour integral with the help of equation (1.9), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting integral with the help of equation (3.1). Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

5. Multivariable I-function

If $\iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \to 1$, the Aleph-function of several variables degenere to the I-function of several variables. The simple integral have been derived in this section for multivariable I-functions defined by Sharma et al [3].

Corollary 1

$$\int_0^\infty \frac{x}{(x^2+y^2)^\rho} ln\left[\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x}\right] \phi_{\mu,\lambda_1,\cdots,\lambda_s}^{(s)}(x_1X_1^{\rho_1}, \cdots, x_sX_s^{\rho_s}, z, a) S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\left(\begin{array}{c} \mathrm{y}_1X^{\gamma_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_tX^{\gamma_t} \end{array}\right)$$

$$\aleph_{u:w}^{0,\mathfrak{n}:v}\left(\begin{array}{c} \mathrm{z}_1X^{\alpha_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_rX^{\alpha_r} \end{array}\right) I_{U:W}^{0,N:V}\left(\begin{array}{c} \mathrm{Z}_1X^{\eta_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_RX^{\eta_R} \end{array}\right) \mathrm{d}x = y^{2-2\rho} \sum_{p_1,\cdots,p_s=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r}$$

$$\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_t=0}^{[N_t/M_t]} \frac{(\mu)_{p_1+\cdots+p_s}}{p_1!\cdots p_s!(a+\lambda_1p_1+\cdots+\lambda_sp_s)^z} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!} G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r}) a_1$$

$$x_1^{p_1}\cdots x_s^{p_s}\, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}} y_1^{K_1} \cdots y_t^{K_t}\, y^{-2(\rho+\sum_{i=1}^{s}\rho_ip_i+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i)} I_{U_{21}:W}^{0,N+2:V}\left(\begin{array}{c} \frac{Z_1}{y^{2\eta_1}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \frac{Z_R}{y^{2\eta_R}} \end{array}\right|$$

$$(2\text{-}\rho-\textstyle\sum_{i=1}^{s}\rho_ip_i-\sum_{i=1}^{t}K_i\gamma_i-\sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i;\eta_1,\cdots,\eta_R),$$

$$\cdots$$

$$\cdots$$

$$(3\text{-}2(\ \rho+\textstyle\sum_{i=1}^{s}\rho_ip_i+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i);2\eta_1,\cdots,2\eta_R)$$

$$\left.\begin{matrix}(3\text{-}\rho-\sum_{i=1}^{s}\rho_ip_i-\sum_{i=1}^{t}K_i\gamma_i-\sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i;\eta_1,\cdots,\eta_R),A:C\\ \cdots\\ \cdots\\ \mathrm{B:D}\end{matrix}\right) \tag{5.1}$$

under the same notationa and conditions that (4.1) with $\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$

6. Aleph-function of two variables

If $s=2$, we obtain the Aleph-function of two variables defined by K.Sharma [5], and we have the following simple integrals.

$$\int_0^{\infty}\frac{x}{(x^2+y^2)^{\rho}}ln\left[\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x}\right]\phi^{(s)}_{\mu,\lambda_1,\cdots,\lambda_s}(x_1X_1^{\rho_1},\cdots,x_sX_s^{\rho_s},z,a)S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t}\left(\begin{matrix}\mathrm{y}_1X^{\gamma_1}\\ \cdots\\ \mathrm{y}_tX^{\gamma_t}\end{matrix}\right)$$

$$\aleph^{0,\mathfrak{n}:v}_{u:w}\left(\begin{matrix}\mathrm{z}_1X^{\alpha_1}\\ \cdots\\ \mathrm{z}_rX^{\alpha_r}\end{matrix}\right)\aleph^{0,N:V}_{U:W}\left(\begin{matrix}\mathrm{Z}_1X^{\eta_1}\\ \cdots\\ \mathrm{Z}_2X^{\eta_2}\end{matrix}\right)\mathrm{d}x=y^{2-2\rho}\sum_{p_1,\cdots,p_s=0}^{\infty}\ \sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\frac{(\mu)_{p_1+\cdots+p_s}}{p_1!\cdots p_s!(a+\lambda_1p_1+\cdots+\lambda_sp_s)^z}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1$$

$$x_1^{p_1}\cdots x_s^{p_s}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}\,y^{-2(\rho+\sum_{i=1}^{s}\rho_ip_i+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i)}\aleph^{0,N+2:V}_{U_{21}:W}\left(\begin{matrix}\frac{Z_1}{y^{2\eta_1}}\\ \cdots\\ \cdots\\ \frac{Z_2}{y^{2\eta_2}}\end{matrix}\ \right|$$

$$(2\text{-}\rho-\textstyle\sum_{i=1}^{s}\rho_ip_i-\sum_{i=1}^{t}K_i\gamma_i-\sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i;\eta_1,\eta_2),$$

$$\cdots$$

$$\cdots$$

$$(3\text{-}2(\ \rho+\textstyle\sum_{i=1}^{s}\rho_ip_i+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i);2\eta_1,2\eta_2)$$

$$\left.\begin{matrix}(3\text{-}\rho-\sum_{i=1}^{s}\rho_ip_i-\sum_{i=1}^{t}K_i\gamma_i-\sum_{i=1}^{\eta}{}_{G_i,g_i}\alpha_i;\eta_1,\eta_2),A:C\\ \cdots\\ \cdots\\ \mathrm{B:D}\end{matrix}\right) \tag{6.1}$$

under the same notation and conditions that (4.1) with $s = 2$

6. I-function of two variables

If $\iota_i, \iota_i', \iota_i'' \rightarrow 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [4] and we obtain the same formula with the I-function of two variables.

Corollary 3

$$\int_0^\infty \frac{x}{(x^2+y^2)^\rho} ln \left[\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x} \right] \phi^{(s)}_{\mu,\lambda_1,\cdots,\lambda_s}(x_1X_1^{\rho_1},\cdots,x_sX_s^{\rho_s},z,a) S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t}\left(\begin{array}{c} \mathrm{y}_1X^{\gamma_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_tX^{\gamma_t} \end{array} \right)$$

$$\aleph^{0,\mathfrak{n}:v}_{u:w}\left(\begin{array}{c} \mathrm{z}_1X^{\alpha_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_rX^{\alpha_r} \end{array} \right) I^{0,N:V}_{U:W}\left(\begin{array}{c} \mathrm{Z}_1X^{\eta_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_2X^{\eta_2} \end{array} \right) \mathrm{d}x = y^{2-2\rho} \sum_{p_1,\cdots,p_s=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r}$$

$$\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_t=0}^{[N_t/M_t]} \frac{(\mu)_{p_1+\cdots+p_s}}{p_1!\cdots p_s!(a+\lambda_1p_1+\cdots+\lambda_sp_s)^z} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!} G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1$$

$$x_1^{p_1}\cdots x_s^{p_s}\, z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}} y_1^{K_1}\cdots y_t^{K_t}\, y^{-2(\rho+\sum_{i=1}^s \rho_ip_i+\sum_{i=1}^t K_i\gamma_i+\sum_{i=1}^r \eta_{G_i,g_i}\alpha_i)} I^{0,N+2:V}_{U_{21}:W}\left(\begin{array}{c} \frac{Z_1}{y^{2\eta_1}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \frac{Z_2}{y^{2\eta_2}} \end{array} \right|$$

$$\left. \begin{array}{c} (2\text{-}\rho - \sum_{i=1}^s \rho_ip_i - \sum_{i=1}^t K_i\gamma_i - \sum_{i=1}^\eta {}_{G_i,g_i}\alpha_i;\eta_1,\eta_2), \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ (3\text{-}2(\ \rho + \sum_{i=1}^s \rho_ip_i + \sum_{i=1}^t K_i\gamma_i + \sum_{i=1}^\eta {}_{G_i,g_i}\alpha_i);2\eta_1,2\eta_2) \end{array} \right)$$

$$\left. \begin{array}{c} (3\text{-}\rho - \sum_{i=1}^s \rho_ip_i - \sum_{i=1}^t K_i\gamma_i - \sum_{i=1}^\eta {}_{G_i,g_i}\alpha_i;\eta_1,\eta_2), A : C \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{B : D} \end{array} \right) \qquad (6.1)$$

under the same notationa and conditions that (4.1) with $s = 2$ and $\iota_i, \iota_i', \iota_i'' \rightarrow 1$

8. Conclusion

In this paper we have evaluated a finite integral involving the multivariable Aleph-functions, a class of polynomials of several variables and the general of sequence of functions.The integral established in this paper is of very general nature as it contains Multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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