More on Minimal δ-Open (Maximal δ-Closed) and δ-Semi-minimal Open (δ-Semi maximal Closed) Sets in Topological Spaces

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Abstract— *In 2008, Caldas M, Jafari S. and Noiri T. [7] introduced the concept of maximal δ-open sets, minimal δ-closed sets, δ-semi-maximal open and δ-semi-minimal closed sets in general topological settings. In the present paper a new class of sets called minimal δ-open sets and maximal δ-closed sets in a topological space are introduced which are the δ-open sets and δ-closed sets respectively. The complement of minimal δ-open set is a maximal δ-closed set. Some properties of δ-semi maximal closed sets, δ-semi minimal open sets are studied.*

Keywords — *Minimal δ-open set, Maximal δ-closed set, δ-semi-minimal open set, δ-semi-maximal closed set.*

I. INTRODUCTION

In 1970, Norman Levine [5] introduced the notion of generalized closed sets and its dual open sets in topological spaces. After him many authors concentrated in this directions and defined various types of generalized closed sets in that spaces. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of δ-open and θ-open sets introduced by N. V. Velicko [3] in 1968. Since the collection of θ-open sets in a topological space (X,τ) forms a topology τ_θ on X, then the union of two θ-open sets is of course θ-open. Moreover, $\tau = \tau_\theta$ if and only if (X,τ) is regular. F. Nakaoka and N. Oda in [1] and [2] introduced the notion of maximal open sets and minimal closed sets. In 2008, Caldas M, Jafari S. and Noiri T. [7] introduced the concept of maximal δ-open sets, minimal δ-closed sets, δ-semi-maximal open and δ-semi-minimal closed sets in general topological settings. In the present paper a new class of sets called minimal δ-open sets and maximal δ-closed sets in a topological space are introduced which are the δ-open sets and δ-closed sets respectively. The complement of minimal δ-open set is a maximal δ-closed set. Some properties of δ-semi maximal closed sets, δ-semi minimal open sets are studied.

II. PRELIMINERY NOTE

For a subset A of a topological space (X,τ), $\delta Cl(A)$ denotes the δ-closure of A with respect to the topological space (X,τ). Sometimes the topological space (X,τ) is simply denoted by X. By a proper open set of a topological space X, we mean an open set $G \neq \varphi$, X and by a proper closed set, we mean a closed set $E \neq \varphi$, X. Throughout the work ordered pairs (X,τ) and (Y,σ) (or X and Y) will denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X. We denote the interior and the closure of a set A by Int(A) and Cl(A), respectively. For a topological space (X,τ) and $A \subset X$, we write $(A,\tau_{/A})$ to denote the subspace on A of (X,τ). Let us recall the following definitions which are useful in the sequel:

Definition 2.1[1]: A proper non-empty δ-open subset U of a topological space X is said to be minimal δ-open set if any δ-open set which is contained in U is φ or U.

Definition 2.2 [2]: A proper non-empty δ-open subset U of a topological space X is said to be maximal δ-open set if any δ-open set containing in U is X or U.

Definition 2.3 [6]: A proper non-empty δ-closed subset F of a topological space X is said to be minimal δ-closed set if any δ-closed set which is contained in F is φ or F.

Definition 2.4 [6]: A proper non-empty δ-closed subset F of a topological space X is said to be maximal δ-closed set if any δ-closed set containing in F is X or F.

Definition 2.3 [3]: Let A be a subset of a space X. A point $x \in X$ is called a δ-cluster point of A if $A \cap U \neq \varphi$, for every regular open set U of X containing x. The set of all δ-cluster points of A is called the δ-closure of A, denoted by $Cl_\delta(A)$.

Definition 2.4 [3]: A subset A of X is called δ-closed if $A = Cl_\delta(A)$. The complement of a δ-closed set is called δ-open set in X.

We denote the collection of all δ-open (respectively, δ-closed) sets by δ-O(X,τ) (respectively, δ-C(X,τ)). It is very well known that the families of all δ-open (resp. θ-open) subsets of (X,τ) are topologies on X which we shall denote by τ_δ (resp.τ_θ). From the definitions it follows immediately that $\tau_\theta \subseteq \tau_\delta \subseteq \tau$. The space (X, τ_δ) is also called the semi-regularization of (X,τ). A space (X,τ) is said to be semi-regular if $\tau_\delta = \tau$ and (X,τ) is regular iff $\tau_\theta = \tau$. It is easily seen that one always has, $A \subseteq Cl(A) \subseteq Cl_\delta(A) \subseteq Cl_\theta(A) \subseteq \bar{A}^\theta$, where, $\bar{A}^\theta$ denotes the closure of A with respect to (X, τ_θ).

III. MINIMAL δ -OPEN AND MAXIMAL δ-CLOSED SETS

This section introduces the notion of maximal δ-closed set and minimal δ-open sets and investigate some of their properties with examples.

Definition 3.1: A non-empty proper δ-open subset U of a topological space X is said to be a minimal δ-open set if any δ-open set contained in U is either φ or U.

Example 3.2: Let X = {a, b, c, d} and topology on X be τ ={X, φ, {a}, {b}, {a, b}}.
δ-open sets are X, φ, {a}, {b} and {a, b}.
Minimal δ-open sets are {a} and {b}.
Maximal δ-closed sets are {b,c,d} and {a,c,d}.

Theorem 3.3. If there is a nonempty finite δ-open set G, then G contains at least one minimal δ-open set.

Proof: Let G be a non-empty finite δ-open set. If G is a minimal δ-open set, then the statement holds true. If G is not a minimal δ-open set, then there exists a δ-open set V_1 such that $\varphi \neq G_1 \subset G$. If G_1 is a minimal δ-open set, then the statement holds true. If G_1 is not a minimal δ-open set, then there exists a δ-open set G_2 such that $\varphi \neq G_2 \subset G_1$. Continuing this process we have a sequence of δ-open sets $G_k \subset G_{k-1} \subset \ldots \subset G_3 \subset G_2 \subset G_1 \subset G$. Since G is a finite set, this process repeats only finitely and so finally we get a minimal δ-open set G_n, for some positive integer n such that $G_n \subset G$.

Corollary 3.4. If G be a finite minimal open set, then there exists at least one minimal δ-open set U such that U⊂G.

Proof: If G is a finite minimal open set, then G is a nonempty finite δ-open set and so by **theorem 3.3** there exists at least one minimal δ-open set U such that U ⊂ G.

Definition 3.5: A non-empty proper δ-closed subset F of a topological space X is said to be a maximal δ-closed set if any δ-closed set which contains F is either X or F.

Example 3.6: Let X = {a, b, c, d} and topology on X be τ ={X, ϕ, {c}, {c,d}, {a,b},{a,b,c}}. Maximal closed sets are {c,d} and {a,b,d}.
δ-closed sets are X, φ, {c,d} and {a, b}.
Maximal δ-closed sets are {a,b} and {c,d}and Minimal δ-open sets are {a,b} and {c,d}.

Theorem 3.5: A non-empty proper subset F of a topological space X is a maximal δ-closed set if and only if X-F is a minimal δ-open set.

Proof: Suppose that F is a maximal δ-closed set and X-F is not a minimal δ-open set. Then there exists a δ-open set U ≠ X−F such that φ ≠ U ⊂ X-F that is F⊂ X-U and X-U is a δ-closed set which is a contradiction to our assumption that F is a Maximal δ- closed set. Hence F is maximal δ-closed set.
Conversely, Suppose X-F is a minimal δ-open set and F is not a Maximal δ-closed set. Then there exists a δ-closed set E other than F and X such that F⊂ E that is proper φ ≠ X-E ⊂ X-F and X-E is a δ-open set. This contradicts our assumption that X-F is a minimal δ-open set. Therefore F is a Maximal δ-closed set.

Theorem 3.6: If F is a non-empty proper co-finite δ-closed subset of a topological space X, then there exists a co-finite maximal δ-closed set E such that F ⊂ E.

Proof: Let F is a non-empty proper co-finite δ-closed set in a topological space X. Then X-F is a non-empty finite δ-open set. This implies that there is a finite minimal δ- open set U such that U ⊂ X-F by theorem 3.3. So, there exists a co-finite maximal closed set E = X-U such that F⊂E.

IV.δ-SEMI-MINIMAL OPEN AND δ-SEMI-MAXIMAL CLOSED SETS

This section introduces the notion of δ-semi-minimal open set and δ-semi-maximal closed sets and investigate some of their properties with examples.

Definition 4.1: A subset A in a topological space X is said to be δ-semi-minimal open set if there exists a minimal open set M Such that M ⊂ A ⊂ Cl(M).
The family of all δ-semi-minimal open sets in a topological space X is denoted by $S\delta M_iO(X)$.

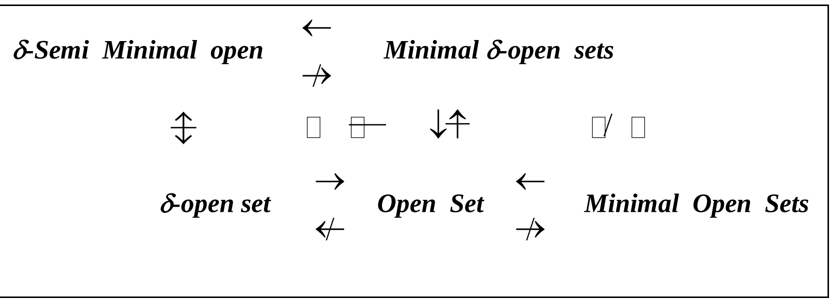
Example 4.2: Let X = {a,b,c,d} and τ = { X,φ,{a},{b},{a,b}} be a topology on X.
Here, δ-open sets are X, φ, {a}, {b} and {a, b}.
δ-semi-minimal open sets are X, φ,{a},{b},{a,d},{b,c},{a,c}, {b,d},{b,c,d} and {a,c,d}.

Example 4.3: In example 3.2 shows that every minimal δ-open set is δ-open set and every δ-open set is open sets but {a, b} is both open and δ-open which is neither minimal δ-open nor δ-Semi minimal open set. Again, {a,c,d} is δ-Semi minimal open which is neither minimal δ-open nor δ-open set.

Example 4.5: Let X = {a, b, c, d} and topology on X be τ ={X, ϕ, {c}, {c,d}, {a,b},{a,b,c}}. Minimal open sets are {c} and {a,b}.
δ-open sets are X, φ, {c,d} and {a, b}.

Minimal δ-open sets are {a,b} and {c,d}.
Here we find that **(1)** {a,b,c} is open which is neither δ-open nor minimal open set
(2) {c} is minimal open which is not minimal δ-open and {c,d} is minimal δ-open but not minimal open set.

Remark 4.6: The above two examples show that **(i)** Minimal δ-open sets are minimal open, δ-open and open sets but not conversely **(ii)** minimal open sets and δ-open sets are each open set but not conversely **(iii)** δ-Semi minimal open and δ-open sets are independent.
The above results are shown in the following diagram.



Theorem 4.7: If G is a δ-semi minimal open set in a topological space X and G ⊂ H ⊂ Cl(M), then H is also δ-semi-minimal open in X.

Proof: Let G be a δ-semi-minimal open in X. Then by definition 4.1 there exists a minimal δ-open set U in X such that U ⊂ G ⊂ Cl(U). Since G ⊂ Cl(U), it follows that Cl(G) ⊂Cl (Cl(U)) = Cl(U). But from hypothesis H ⊂ Cl(G), therefore, it follows that U ⊂ H ⊂ Cl(U). Therefore by definition 4.1 it follows that H is δ-semi-minimal open in X.

Theorem 4.8: Let X be a topological space and $M_i\delta O(X)$ be the class of all minimal δ-open sets in X. The following results hold good.

(i) $M_i\delta O(X) \subseteq \delta SM_iO(X)$

(ii) If $G \in \delta SM_iO(X)$ and $G \subseteq H \subseteq Cl(M)$, then $H \in \delta SM_iO(X)$.

Proof: This follows from theorem 4.7.

Theorem 4.9: Let Y be a subspace of a topological space X and G be a subset of Y. If G is δ-semi-minimal open in X then G is δ-semi-minimal open in Y.

Proof: Suppose G is δ-semi-minimal open in X. By definition 4.1 there exists a minimal δ-open set U in X such that U ⊂ G ⊂ Cl(U). Now U ⊂ G ⊂ Y. Hence U ∩Y=U. Since U is minimal δ-open in X. U∩Y=U is minimal δ-open in Y. Now we have U ⊂ G ⊂ Cl(U). Therefore, U∩Y ⊂ G∩Y ⊂ Y∩Cl(U) which implies that U ⊂ G ⊂ $Cl_Y(U)$. Thus there exists a minimal δ-open set U in Y such that U ⊂ G ⊂ $Cl_Y(U)$. Therefore by definition 4.1 it follows that G is δ-semi-minimal open in Y.

Theorem 4.10: If M and N are minimal δ-open sets in a topological space X and U⊂X such that N⊂U⊂Cl(N). If M ∩ N = φ then U ∩ W= φ.

Proof: Since M ∩ N = φ, it follows that N⊂X-M. Therefore Cl(N) ⊂ Cl(X-M) = X-M. Since X-M is maximal δ-closed set and every maximal δ-closed set is closed set. Also we have N⊂U⊂Cl(N). Therefore, U ⊂ Cl(N) ⊂ X-M. Thus U ⊂ X-M which means U ∩W= φ.

Remark 4.11: Intersection and union of any two δ-semi-minimal open sets need not be δ-semi-minimal open. It can be shown from the following example:

Example 4.12: Let X = {a,b,c,d} and τ = { X,φ,{a},{b},{a,b}} be a topology on X.
Here, δ-open sets are X, φ, {a}, {b} and {a, b}.
δ-semi-minimal open sets are X, φ,{a},{b},{a,d},{b,c},{a,c}, {b,d}, {b,c,d} and {a,c,d}.
Clearly, {b,c} and {a,c} are δ-semi minimal open sets and {b,c}∩{a,c}={c} which is not a δ-semi-minimal open set. Again {a} and {b} are δ-semi minimal open sets and {a}∪{b}={a,b} which is not a δ-semi-minimal open set.

Definition 4.13: A subset F of a topological space X is said to be δ-semi maximal closed set if X-F is δ-semi-minimal open set.
The family of all δ-semi-maximal closed sets in a topological space X is denoted by $\delta SM_iC(X)$.

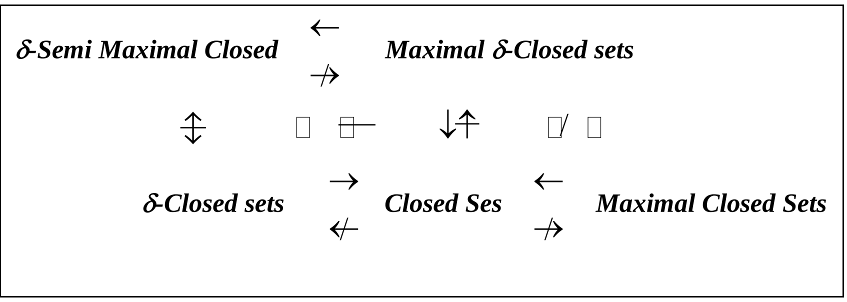
Example 4.14: Let X = {a,b,c,d}, τ = { X,φ,{a},{b},{a,b}} be a topological space.
Closed sets are X, φ,{b,c,d},{a,c,d} and {c,d}.
Maximal δ-Closed sets are {b,c,d} and {a,c,d}
δ-Semi-Maximal Closed Sets are {a},{b},{b,c},{b,d},{a,c}, {a,d},{b,c,d} and {a,c,d} and δ-Semi-Maximal Closed Sets are {a},{b},{b,c},{b,d},{a,c}, {a,d},{b,c,d} and {a,c,d}.

Example 4.15: In example 3.6 we find that (1) {d} is closed which is neither δ-closed nor maximal closed set.
(2) {a,b,d} is maximal closed which is not maximal δ-closed and {a,b} is maximal δ-closed but not maximal closed set.

Example 4.16: Let X = {a, b, c, d} and topology on X be τ ={X, φ, {a}, {b}, {a, b}}.
δ-closed sets are X, φ, {c,d} and {a, b}.
Maximal δ-closed sets are {a,b} and {c,d}.
This example shows that every maximal δ-closed set is δ-closed sets and every δ-closed set is closed set but {c,d} is both closed and δ-closed which is neither maximal δ-closed nor δ-Semi maximal closed set. Again {b} is δ-semi maximal closed which is neither δ-closed nor maximal δ-closed set.

Remark 4.17: The above two examples show that **(i)** Maximal δ-closed sets are maximal closed, δ-closed, closed and δ-semi maximal closed sets but not conversely **(ii)** Maximal closed sets and δ-closed sets are each closed set but not conversely **(iii)** δ-

Semi Maximal Closed and δ-Closed sets are independent.
The above results are shown in the following diagram.



Theorem 4.18: A subset W of a topological space X is δ-semi-maximal closed iff there exists a maximal δ-closed set N in X such that Int(N)⊂W⊂N.

Proof: Suppose W is a δ-semi-maximal closed in X then by definition 4.13 X-W is δ-semi-minimal open in X. Therefore by definition 4.1, there exists a minimal δ-open set M such that M ⊂ X-W ⊂ Cl(M) which implies that X-Cl(M) ⊂ X-(X-W) ⊂ X-M which implies X-Cl(M)⊂W⊂ X-M. But it is know that X-Cl(M) = Int(X-M) take X-M=N so, that N is a maximal δ-closed set such that Int(N) ⊂W ⊂N.

Conversly, suppose that there exist a maximal δ-closed set N in X such that int(N) ⊂W ⊂N. Therefore, it follows that X-N ⊂ (X-W) ⊂ X-Int(N). But it is know that X-int(N)) = Cl(X-N). Therefore there exists a minimal δ-open set X-N such that X-N ⊂ X-W ⊂ Cl(X-N). Thus by definition 4.1 it follows that X-W is δ-semi-minimal open in X. Hence by definition 4.13 it follows that W is δ-semi-maximal closed set.

Theorem 4.19: If H is δ-semi-maximal closed in X and Int(H)⊂W⊂H then W is δ-semi-maximal closed in X.

Proof: Let H be δ-semi-maximal closed in X then by definition of δ-semi-maximal closed sets there exists a maximal δ-closed set F such that Int(F) ⊂ H ⊂F. Now Int(F)⊂H which implies Int(F) =Int(Int(F)) ⊂Int(N). But, Int(H) ⊂W, we have Int(F) ⊂W. Further, since Int(F) ⊂Int(H) ⊂W ⊂ H ⊂ F. It follows that Int(F) ⊂ W ⊂ F. Thus there exists a maximal δ-closed set F such that Int(F) ⊂W⊂ F. Therefore W is δ-semi-maximal closed in X.

Theorem 4.20: The following three properties of a subset F of a topological space X are equivalent.

(a) F is δ-semi-maximal closed set in X.
(b) Int(Cl(F)) ⊂ F.
(c) (X-F) is δ-semi-minimal open set in X.

Remark 4.21: Intersection and union of any two δ-semi-maximal closed sets need not be δ-semi-maximal closed set. It can be shown from the following example:

Example 4.22: In example 4.12, we find that δ-open sets are X, φ, {a}, {b} and {a, b}.

δ-semi-maximal closed sets are X, φ,{a},{b},{a,d},{b,c},{a,c}, {b,d},{b,c,d} and {a,c,d}.

Clearly, {b,c,d} and {a,c,d} are δ-semi-maximal closed sets and {b,c,d}∩{a,c,d}={c,d} which is not a δ-semi-minimal open set. Again {a,c} and {b,c} are δ-semi-maximal closed sets and {a,c}∪{b,c}={a,b,c} which is not a δ-semi-maximal closed set.

V. Conclusions

Topology plays an significant role in space time geometry, quantum physics, high energy physics and superstring theory. So the aim of this paper is devoted to study topological spaces. In this work, the concept of maximal δ-closed sets, minimal δ-open sets, δ-semi maximal closed and δ-semi minimal open sets which are fundamental results for further research on topological spaces are introduced and aimed in investigating the properties of these new notions of open sets. Interrelationships among these new concepts with existing sets are also examined with the help of example and counter examples. Hope that the findings in this work will help researcher enhance and promote the further study on topological spaces to carry out a general framework for their applications in separation axioms, connectedness, compactness etc. and also in practical life.

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