A NOTE ON (α, β)-$*$-i-n-DERIVATIONS IN RINGS WITH INVOLUTION

Mohammad Aslam Siddeeque

Department of Mathematics
Aligarh Muslim University
Aligarh -202002(India)

Abstract: Let R be a ring with involution '$*$'. In this paper we introduce the notion of (α, β)-$*$-i-n-derivation in R. An additive mapping $x \mapsto x^*$ of R into itself is called an involution on R if it satisfies the conditions; (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring R equipped with an involution '$*$' is called a $*$-ring. In the present paper it is shown that if a $*$-prime ring R admits a nonzero (α, β)-$*$-i-n-derivation D, then R is commutative. Further an important property of (α, β)-$*$-i-n-derivation in semiprime $*$-ring has also been derived.

1. INTRODUCTION

Throughout the paper, R will represent an associative ring with center Z. Ring R is called prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. It is called semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $d : R \longrightarrow R$ is said to be a (α, β)-derivation on R if there exist endomorphisms α and β of R such that $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. An additive mapping $x \mapsto x^*$ of R into itself is called an involution on R if it satisfies the conditions; (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring R equipped with an involution '$*$' is called a $*$-ring. A ring R with involution '$*$' is said to be $*$-prime if $aRb = aRb^* = \{0\}$, (equivalently $aRb = a^*Rb = \{0\}$) where $a, b \in R$ implies that either $a = 0$ or $b = 0$. It is to be noted that every prime ring having an involution '$*$' is $*$-prime but the converse is not true

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in general. Of course, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x,y) = (y,x)$, is $*_{ex}$-prime but not prime.

Let R be a $*$-ring. An additive mapping $d : R \longrightarrow R$ is said to be a $*$-derivation on R if $d(xy) = d(x)y^* + xd(y)$ holds for all $x, y \in R$. If R is a commutative $*$-ring, then $d : R \longrightarrow R$ defined by $d(x) = a(x - x^*)$, where $a \in R$, is a $*$-derivation on R (for reference see [10]). An additive map $T : R \longrightarrow R$ is called a left (resp. right) $*$-multiplier if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^*T(y)$) holds for all $x, y \in R$. An additive mapping $d : R \longrightarrow R$ is said to be a (α, β)-$*$-derivation on R if $d(xy) = d(x)\alpha(y^*) + \beta(x)d(y)$ holds for all $x, y \in R$. There has been a great deal of work concerning commutativity of prime and semiprime rings admitting different types of derivations (for reference see $[3-6]$,[10],[14] etc., where further references can be found). Very recently Ali and Khan [2] defined symmetric $*$-biderivation, symmetric left (resp. right) $*$-bimultiplier and studied some properties of prime $*$-rings and semiprime $*$-rings, admitting symmetric $*$-biderivation and symmetric left (resp. right) $*$-bimultiplier. Motivated by these concepts the authors [6] introduced the notion of $*$-n-derivations and $*$-n-multipliers in $*$-rings and studied these notions in the setting of prime $*$-rings and semiprime $*$-rings.

Let R be a $*$-ring and n be any fixed positive integer. An n-additive (i.e.; additive in each argument) mapping $D : R^n \longrightarrow R$ is called a $*$-n-derivation of R if the relations

$$\begin{aligned} D(x_1, x_2, \cdots, x_{i-1}, x_i x_i', x_{i+1}, \cdots, x_n) = \; & D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)(x_i')^* \\ & + x_i D(x_1, x_2, \cdots, x_{i-1}, x_i', x_{i+1}, \cdots, x_n) \end{aligned}$$

hold for all $x_1, x_2, \cdots, x_{i-1}, x_i, x_i', x_{i+1}, \cdots, x_n \in R$, $i = 1, 2, 3, \cdots, n$ (See [6] for further reference). We have a weaker family of derivations in $*$-ring R also. Of course this family generalizes the notions of $*$-n-derivations discussed above. Let n be a fixed positive integer and i be an integer with $1 \leq i \leq n$. An n-additive (i.e.; additive in each argument) mapping $D : R^n \longrightarrow R$ is called a $*$-i-n-derivation of R if the relation

$$\begin{aligned} D(x_1, x_2, \cdots, x_{i-1}, x_i x_i', x_{i+1}, \cdots, x_n) = \; & D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)(x_i')^* \\ & + x_i D(x_1, x_2, \cdots, x_{i-1}, x_i', x_{i+1}, \cdots, x_n) \end{aligned}$$

holds for all $x_1, x_2, \cdots, x_{i-1}, x_i, x_i', x_{i+1}, \cdots, x_n \in R$. From the above definition it is obvious that if D is a $*$-i-n-derivation of R for each i with $1 \leq i \leq n$, then D is a $*$-n-derivation of R and conversely. It can be also observed that every $*$-n-derivation is a

2

$*$-i-n-derivation but its converse is not true (See [7] for further reference).

Motivated by the notion of (α, β)-$*$-n-derivation in a $*$-ring R (See[8] for further reference), in the present paper, we introduce the notion of (α, β)-$*$-i-n-derivation in a $*$-ring R, which generalizes the notion of $*$-i-n-derivation discussed above. Let n be a fixed positive integer and i be an integer with $1 \leq i \leq n$. An n-additive (i.e.; additive in each argument) mapping $D : R^n \longrightarrow R$ is called a (α, β)-$*$-i-n-derivation of R if the relation

$$\begin{aligned} D(x_1, x_2, \cdots, x_{i-1}, x_i x_i^{'}, x_{i+1}, \cdots, x_n) = \; & D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\alpha((x_i{'})^*) \\ & +\beta(x_i)D(x_1, x_2, \cdots, x_{i-1}, x_i^{'}, x_{i+1}, \cdots, x_n) \end{aligned}$$

holds for all $x_1, x_2, \cdots, x_{i-1}, x_i, x_i^{'}, x_{i+1}, \cdots, x_n \in R$.
Recently Ashraf et.al. [6, Theorem 3.1] proved that if a prime $*$-ring R admits a nonzero $*$-n-derivation D, then R is commutative. We have proved its analogue for (α, β)-$*$-i-n-derivation in the setting of $*$-prime rings which is a larger class of $*$-rings than the class of prime rings with involution. Another property of (α, β)-$*$- i-n-derivation for semiprime $*$-rings has also been derived. We have also proved a result related with the equality of two (α, β)-$*$-n-derivations D_1 and D_2 in prime $*$-rings. In fact, our results generalize, extend, improve and compliment some results obtained earlier on $*$-derivation, $*$-n-derivation, (α, β)-$*$-n-derivation for prime $*$-rings and semiprime $*$-rings in [6, 8].

2. MAIN RESULTS

Throughout the paper unless otherwise stated α and β will represent endomorphisms of R.
Theorem 2.1. Let R be a $*$-prime ring. If it admits a nonzero (α, β)-$*$-i-n-derivation D and α is onto, then R is commutative.

Proof. By hypothesis, for all $x_1, y, z, x_2, \cdots, x_n \in R$, we have

$$\begin{aligned} & D(x_1, x_2, \cdots, x_{i-1}, (x_i y)z, x_{i+1} \cdots, x_n) \\ & = D(x_1, x_2, \cdots, x_{i-1}, (x_i y), x_{i+1} \cdots, x_n)\alpha(z^*) \\ & +\beta(x_i y)D(x_1, x_2, \cdots, x_{i-1}, z, x_{i+1} \cdots, x_n) \end{aligned}$$

$$
\begin{aligned}
&= \{D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\alpha(y^*)\\
&+\beta(x_i)D(x_1, x_2, \cdots, x_{i-1}, y, x_{i+1} \cdots, x_n)\}\alpha(z^*)\\
&+\beta(x_i)\beta(y)D(x_1, x_2, \cdots, x_{i-1}, z, x_{i+1} \cdots, x_n)\\
&= D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\alpha(y^*)\alpha(z^*)\\
&+\beta(x_i)D(x_1, x_2, \cdots, x_{i-1}, y, x_{i+1} \cdots, x_n)\alpha(z^*)\\
&+\beta(x_i)\beta(y)D(x_1, x_2, \cdots, x_{i-1}, z, x_{i+1} \cdots, x_n)
\end{aligned}
$$

Also

$$
\begin{aligned}
&D(x_1, x_2, \cdots, x_{i-1}, x_i(yz), x_{i+1} \cdots, x_n)\\
&= D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1} \cdots, x_n)\alpha(yz)^*\\
&+\beta(x_i)D(x_1, x_2, \cdots, x_{i-1}, yz, x_{i+1} \cdots, x_n)\\
&= D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\alpha(z^*y^*)\\
&+\beta(x_i)\{D(x_1, x_2, \cdots, x_{i-1}, y, x_{i+1} \cdots, x_n)\alpha(z^*)\\
&+\beta(y)D(x_1, x_2, \cdots, x_{i-1}, z, x_{i+1} \cdots, x_n)\}\\
&= D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\alpha(z^*)\alpha(y^*)\\
&+\beta(x_i)D(x_1, x_2, \cdots, x_{i-1}, y, x_{i+1} \cdots, x_n)\alpha(z^*)\\
&+\beta(x_i)\beta(y)D(x_1, x_2, \cdots, x_{i-1}, z, x_{i+1} \cdots, x_n)
\end{aligned}
$$

Combining the above two relations, we get $D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\alpha(y^*)\alpha(z^*) = D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ $\alpha(z^*)\alpha(y^*)$ for all $x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n, y, z \in R$.
Putting y^* and z^* in the places of y and z respectively, we find that $D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\alpha(y)\alpha(z) = D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ $\alpha(z)\alpha(y)$. Since α is onto, we conclude that

$$D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)yz = D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)zy. \quad (3.1)$$

Now replacing y by yr where $r \in R$, in the relation (3.1) and using it again we arrive at $D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)R([r, z]) = \{0\}$. This also gives us $D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)R[r, z]^* = \{0\}$. Since $D \neq 0$, $*$-primeness of R implies that $rz = zr$ for all $z, r \in R$ and hence R is commutative.

Corollary 2.1([6],Theorem 3.1). Let R be a prime $*$-ring. If R admits a nonzero $*$-n-derivation D, then R is commutative.

Corollary 2.2([8],Theorem 2.1). Let R be a $*$-prime ring. If it admits a nonzero $(\alpha, \beta)^*$-n-derivation D such that α is onto then R is commutative.

4

Theorem 2.2. Let R be a semiprime $*$-ring admitting a (α, β)-$*$-i-n-derivation D and α is onto. Then $D(R, R, \cdots, R) \subseteq Z$.

Proof. Since R is a $*$-ring having a (α, β)-$*$-i-n-derivation D and α is onto, we have relation (3.1). Putting $yD(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ in place of y in the relation (3.1) and using it again we get $D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)y[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z] = 0$. This in turn gives the following

$$zD(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)y[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z] = 0. \quad (3.2)$$

Replacing y by zy in the relation $D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)y[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z] = 0$, we obtain that

$$D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)zy[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z] = 0. \quad (3.3)$$

Now comparing the identities (3.2) and (3.3) we arrive at $[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z]y[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z] = 0$ i.e.; $[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z]R[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z] = \{0\}$. Now semiprimeness of R yields that $[D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n), z] = 0$ i.e; $D(R, R, \cdots, R) \subseteq Z$.

Corollary 2.3([6], Theorem 3.15). Let R be a semiprime $*$-ring, admitting a $*$-n-derivation D. Then $D(R, R, \cdots, R) \subseteq Z$.

Corollary 2.4([8], Theorem 2.13). Let R be a semiprime $*$-ring. If R admits a $(\alpha, \beta)^*$-n-derivation D and α is onto, then $D(R, R, \cdots, R) \subseteq Z$.

Theorem 2.3. Let R be a prime ring with involution having, (α, β)-$*$-n-derivations D_1 and D_2. Next assume that $I_1, I_2, \cdots, I_n$ are nonzero ideals of R such that $D_1(i_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n)$ for all $i_p \in I_p, 1 \leq p \leq n$ and β is an automorphism. Then $D_1 = D_2$.

Proof. Since

$$D_1(i_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n) \quad (3.4)$$

for all $i_p \in I_p, 1 \leq p \leq n$. Now putting i_1r_1, where $r_1 \in R$, for i_1 in the relation (3.4) we have $D_1(i_1r_1, i_2, \cdots, i_n) = D_2(i_1r_1, i_2, \cdots, i_n)$ i.e.; $D_1(i_1, i_2, \cdots, i_n)\alpha(r_1^*) + \beta(i_1)D_1(r_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n)\alpha(r_1^*) + \beta(i_1)D_2(r_1, i_2, \cdots, i_n)$. Using the relation (3.4) we get $\beta(i_1)D_1(r_1, i_2, \cdots, i_n) = \beta(i_1)D_2(r_1, i_2, \cdots, i_n)$ i.e.; $\beta(i_1)\{D_1(r_1, i_2, \cdots, i_n) - D_2(r_1, i_2, \cdots, i_n)\} = 0$. This shows that $\beta(i_1)R\{D_1(r_1, i_2, \cdots, i_n) - D_2(r_1, i_2, \cdots, i_n)\} = \{0\}$. As β is one-one, we have $\beta(I_1) \neq \{0\}$, primeness of R implies that

$$D_1(r_1, i_2, \cdots, i_n) = D_2(r_1, i_2, \cdots, i_n) \tag{3.5}$$

for all $r_1 \in R, i_p \in I_p, 2 \leq p \leq n$. Now putting i_2r_2, where $r_2 \in R$, for i_2 in the relation (3.5) we get $D_1(r_1, i_2r_2, \cdots, i_n) = D_2(r_1, i_2r_2, \cdots, i_n)$ i.e.; $D_1(r_1, i_2, \cdots, i_n)\alpha(r_2^*) + \beta(i_2)D_1(r_1, r_2, \cdots, i_n) = D_2(r_1, i_2, \cdots, i_n)\alpha(r_2^*) + \beta(i_2)D_2(r_1, r_2, \cdots, i_n)$. By using relation (3.5) we get $\beta(i_2)D_1(r_1, r_2, i_3 \cdots, i_n) = \beta(i_2)D_2(r_1, r_2, i_3, \cdots, i_n)$ i.e.; $\beta(i_2)\{D_1(r_1, r_2, i_3, \cdots, i_n) - D_2(r_1, r_2, i_3, \cdots, i_n)\} = 0$. This shows that $\beta(i_2)R\{D_1(r_1, r_2, i_3, \cdots, i_n) - D_2(r_1, r_2, i_3, \cdots, i_n)\} = \{0\}$. As β is one-one, we have $\beta(I_2) \neq \{0\}$, primeness of R implies that $D_1(r_1, r_2, i_3, \cdots, i_n) = D_2(r_1, r_2, i_3, \cdots, i_n)$. Now proceeding inductively in the same way as above we conclude that $D_1 = D_2$.

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