On Some identities for Generalized (k, r) Fibonacci Numbers

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Abstract — *We have formulated Binet`s formula for Generalized (k, r) Fibonacci numbers. Also we have obtained some identities, including generating function for (k, r) Fibonacci sequence.*

Keywords — (k, r) Fibonacci numbers, Binet`s formula, Generating function, recurrence relation

1. INTRODUCTION

The Fibonacci number is crowd pleasing topic for mathematical enhancement and popularization. It is notable to many for locking up wondrous and astonishing properties. Fibonacci strikes one as in voluminous mathematical problems. Many researchers have done worthwhile work on this absorbing topic.

The k-Fibonacci number defined by Falcon and Plaza [3] depends only on one integer parameter k which is as follows:
For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\epsilon N}$ is defined recurrently by

$$F_{k,0} = 0\ ,\ F_{k,1} = 1\ ,$$

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad (1)$$

Falcon [6] applied the definition of r-distance to the k- Fibonacci numbers in such a way that it generalized earlier results [1, 2]. Generalized (k, r) Fibonacci numbers are defined as,
The Generalized (k, r) Fibonacci numbers $F_{k,n}(r)$, for $k \geq 1, n \geq 0, r \geq 1$ is

$$F_{k,n}(r) = kF_{k,n-r}(r) + F_{k,n-2}(r) \quad \text{for } n \geq r\ , \quad (2)$$

With the initial condition $F_{k,n}(r) = 1, n = 0, 1, 2, \dots, r-1$ except $F_{k,1}(1) = k.$
The characteristic equation, associated to the recurrence relation (2)

$$\alpha^2 - k\alpha - 1 = 0, \quad (3)$$

with two distinct roots α_1 and α_2 , we get

$$\alpha_1 = \frac{k+\sqrt{k^2+4}}{2}, \alpha_2 = \frac{k-\sqrt{k^2+4}}{2},$$

$$\alpha_1 + \alpha_2 = k,$$

$$\alpha_1 . \alpha_2 = -1.$$

2. *Properties of Generalized (k, r) Fibonacci Numbers*

2.1. First Explicit Formula for Generalized (k, r) Fibonacci *Numbers*

The French mathematician Binet concocted two conspicuous analytical formulas for the Fibonacci and Lucas numbers, in 19th century. In our context, we can articulate the Binet`s formula for (k, r) Fibonacci numbers from the equation (3) with two distinct roots α_1 and α_2.

***Proposition 1. (Binet`s formula*)**

The n^{th} (k, r) Fibonacci number is given by

$$F_{k,n}(r) = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}. \quad (4)$$

Proof.
Since the equation (3) has two distinct roots, the sequence

$$F_{k,n}(r) = C_1(r_1)^n + C_2(r_2)^n \quad (5)$$

is the solution of equation (2). By giving to n the values n = 0 and n = 1, and r = 1 and solving linear equation, we obtain a unique values

$$C_1 = \frac{k+\sqrt{k^2+4}}{2\sqrt{k^2+4}}, C_2 = \frac{\sqrt{k^2+4}-k}{2\sqrt{k^2+4}},$$

Using (5), we get

$$F_{k,n}(r) = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}.$$

In this paper, we present properties of (*k*, r) Fibonacci numbers like Catalan's identity, Cassini's identity and d'ocagnes's Identity.

2.2. *Catalan`s Identity*

A Belgian mathematician Eugene Charles Catalan who worked for science Belgian academy of formed catalan`s identity for Fibonacci number, now we are acquainting the Catalan`s identity for (k, r) Fibonacci number.

***Proposition 2. (Catalan`s Identity*)**

$$F_{k,n-m}(r).F_{k,n+m}(r) - F_{k,n}^2(r) = (-1)^{n+1}F_{k,m}^2(r) \quad (6)$$

Proof.
Using the Binet`s formula (4) and we get

$$F_{k,n-m}(r).F_{k,n+m}(r) - F_{k,n}^2(r) = \left(\frac{\alpha_1^{n-m+1} - \alpha_2^{n-m+1}}{\alpha_1 - \alpha_2}\right).\left(\frac{\alpha_1^{n+m+1} - \alpha_2^{n+m+1}}{\alpha_1 - \alpha_2}\right) - \left(\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}\right)^2$$

=

$$\frac{\alpha_1^{2n+2}-\alpha_1^{n-m+1}.\alpha_2^{n+m+1}-\alpha_2^{n-m+1}.\alpha_1^{n+m+1}+\alpha_2^{2n+2}-\alpha_1^{2n+2}-\alpha_2^{2n+2}+2\alpha_1^{n+1}.\alpha_2^{n+1}}{(\alpha_1-\alpha_2)^2}$$

$$= \frac{-\alpha_1^{n-m+1}.\alpha_2^{n+m+1}-\alpha_2^{n-m+1}.\alpha_1^{n+m+1}+2\alpha_1^{n+1}.\alpha_2^{n+1}}{(\alpha_1-\alpha_2)^2}$$

$$= \left(\frac{\alpha_1^{n-m+1}.\alpha_2^{n+m+1}-\alpha_2^{n-m+1}.\alpha_1^{n+m+1}+2\alpha_1^{n+1}.\alpha_2^{n+1}}{(\alpha_1-\alpha_2)^2}\right)$$

$$= -(\alpha_1.\alpha_2)^n\left(\frac{\alpha_1^{2m+1}+\alpha_2^{2m+1}-2\alpha_1^{m+1}.\alpha_2^{m+1}}{(\alpha_1-\alpha_2)^2}\right)$$

$$= -(-1)^n\left(\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2}\right)^2$$

$$= (-1)^{n+1}F_{k,m}^2(r).$$

2.3. ***Cassini`s Identity***

Here is Cassini Identity devised by a French astronomer Jean Dominique Cassini in 1680:

Proposition 3. (Cassini`s Identity)

$$F_{k,n-1}(r).F_{k,n+1}(r) - F_{k,n}^2(r) = (-1)^{n+1}F_{k,1}^2(r) \quad (7)$$

Proof.

In Catalan`s identity, taking m = 1, the proof is accomplished.

2.4. ***d`Ocagne`s Identity***

Proposition 4. (d`Ocagne`s Identity)

If m > n, then

$$F_{k,m}(r).F_{k,n+1}(r) - F_{k,m+1}(r).F_{k,n}(r) = (-1)^{n+1}F_{k,m-n-1}(r).$$

(8)

Proof.

Using the Binet`s formula (4) and m > n, we get

$$F_{k,m}(r).F_{k,n+1}(r) - F_{k,m+1}(r).F_{k,n}(r) = \left(\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1^{n+2}-\alpha_2^{n+2}}{\alpha_1-\alpha_2}\right) - \left(\frac{\alpha_1^{m+2}-\alpha_2^{m+2}}{\alpha_1-\alpha_2}\right).\left(\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}\right)$$

$$=(\alpha_1\alpha_2)(\alpha_1\alpha_2)^n\left(\frac{\alpha_1^{m-n}-\alpha_2^{m-n}}{\alpha_1-\alpha_2}\right)$$

$$=(-1)^{n+1}F_{k,m-n-1}(r).$$

2.5. Limit of the Quotient of Two Consecutive Terms

From these sequences we get the limit of the quotient of two consecutive terms, which is equal to the positive root of corresponding characteristic equation.

Proposition 5.

$$\lim_{n\to\infty}\frac{F_{k,n}(r)}{F_{k,n-1}(r)} = \alpha_1.$$

(9)

Proof.

We have

$$\lim_{n\to\infty}\frac{F_{k,n}(r)}{F_{k,n-1}(r)} = \lim_{n\to\infty}\left(\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1-\alpha_2}{\alpha_1^n-\alpha_2^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1^n-\alpha_2^n}\right)$$

since $\alpha_2 < \alpha_1$, then $\lim_{n\to\infty}\left(\frac{\alpha_2}{\alpha_1}\right)^n = 0$,

we get , $\lim_{n\to\infty}\frac{F_{k,n}(r)}{F_{k,n-1}(r)} = \alpha_1$.

Proposition 6.

$$\lim_{n\to\infty}\frac{F_{k,n-1}(r)}{F_{k,n}(r)} = \frac{1}{\alpha_1}.$$

(10)

Proof.

We can also show this like Proposition 5.

2.7. ***Generating function for the (k, r)-Generalized Fibonacci Number:***

The following paragraph explains the generating function for (k, r) Fibonacci numbers**.**

Proposition 7.

Generating function of sequence of $F_k(r) = \{F_{k,n}(r)\}$ is given by

$$G(F_{k,n}(r):x) = \frac{1+x}{1-x^2-kx^r}.$$

(11)

Proof.

$$G(F_{k,n}(r):x) = \sum_{n=0}^{\infty}F_{k,n}(r)x^n$$

$$= F_{k,0}(r)x^0 + F_{k,1}(r)x^1 + F_{k,2}(r)x^2 + F_{k,3}(r)x^3 + \cdots$$

$$= 1 + x + \sum_{n=2}^{\infty}F_{k,n}(r)x^n$$

$$= 1 + x + \sum_{n=2}^{\infty}[kF_{k,n-r}(r)+F_{k,n-2}(r)]x^n$$

$$= 1 + x + kx^r\sum_{n=2}^{\infty}kF_{k,n-r}(r)x^{n-r} + x^2\sum_{n=2}^{\infty}F_{k,n-2}(r)x^{n-2}$$

$$= 1 + x + kx^r\sum_{j=0}^{\infty}kF_{k,p}(r)x^p + x^2\sum_{j=o}^{\infty}F_{k,j}(r)x^j$$

where p = n – r, n = 0, 1, 2, 3, … and r = 1, 2, 3, …and j = n – 2

so $(1 - kx^r - x^2)\sum_{n=0}^{\infty}F_{k,n}(r)x^n = 1 + x$

we have $F_{k,n}(r) = \frac{1+x}{1-x^2-kx^r}$.

3. Conclusion

In this paper we have knocked down the Binet`s formula for (k, r) Generalized Fibonacci numbers. Finally we have opened gets to view the properties like Catalan’s identity, Cassini’s identity or Simpson’s identity and d’ocagnes’s identity for generalized (k, r) Fibonacci numbers.

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