Generalized Elliptic-Type Integrals and Generating Functions with Multivariable Aleph-Function

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ABSTRACT
Elliptic-type integrals have their importance and potential in certain problems in radiation physics and nuclear technology [4,5,7,10,15,17,22,23]. A number of earlier works on the subject contains several interesting unifications and generalizations of some significant families of elliptic-type integrals. The present paper is intended to obtain certain new theorems on generating functions. The results obtained in this paper are of manifold generality and basic in nature. Beside deriving various known and new elliptic-type integrals and their generalizations these theorems can be used to evaluate various Euler-type integrals involving a number of generating functions.

Keywords: Elliptic-type integrals, Euler-type integrals, Generating functions, multivariable Aleph-function.

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1. Introduction.

$$\Omega_j(k) = \int_0^{\pi} (1-k^2\cos\theta)^{-j-\frac{1}{2}}\mathrm{d}\theta, \tag{1.1}$$

where $j = 0,1,2,\cdots$ and $0 \leqslant k \leqslant 1$ was studied by Epstein-Hubbell [12], for the first time. Due to its occurrence in a number of physical problems [4,5,13, 15, 22, 23, 28], in the form of single and multiple integrals, several authors notably Kalla [16, 17] and Kalla et al. [18], Kalla and Al-Saqabi [19], Kalla et al. [20], Salman [25], Saxena et al. [28] and Srivastava and Bromberg [36], have investigated various interesting unifications (and generalizations) of the elliptic-type integrals (1.1). Some of the important generalizations of elliptic-type integral (1.1) are as follows:
Kalla [16, 17] introduced the generalization of the form:

$$R_\mu(k,\alpha,\gamma) = \int_0^{\pi} \frac{\cos^{2\alpha-1}(\theta/2)sin^{2\gamma-2\alpha-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\mathrm{d}\theta, \tag{1.2}$$

where $0 \leqslant k \leqslant 1, Re(\gamma) > Re(\alpha) > 0, Re(\mu) > -\frac{1}{2}$.

Results for this generalization are also derived by Glasser and Kalla [14].

Al-Saqabi [1] defined and studied the generalization given by the integral

$$B_\mu(k,m,\upsilon) = \int_0^{\pi} \frac{\cos^{2m}(\theta)sin^{2\upsilon}(\theta)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\mathrm{d}\theta, \tag{1.3}$$

where $0 \leqslant k \leqslant 1\ m \in \mathbb{N}_0, \mu \in \mathbb{C}, Re(\mu) > -\frac{1}{2}$.

Asymptotic expansion of (1.3) has recently been discussed by Matera et al. [24]. The integral

$$A_\upsilon(\alpha,k) = \int_0^{\pi} \frac{exp\left(\alpha sin^2(\theta/2)\right)}{(1-k^2\cos\theta)^{\upsilon+\frac{1}{2}}}\mathrm{d}\theta, \tag{1.4}$$

where $0 \leqslant k \leqslant 1$, $\alpha,\upsilon \in \mathbb{R}$; presents another generalization of (1), given by Siddiqi [33].

Srivastava and Siddiqi [35] have given an interesting unification and extension of the families of elliptic- type integrals in the following form:

$$A_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k)=\int_0^{\pi}\frac{\cos^{2\alpha-1}(\theta/2)sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\left[1-\rho sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda}\mathrm{d}\theta, \tag{1.5}$$

where $0\leqslant k\leqslant 1, Re(\alpha)>0, Re(\beta)>0, \lambda,\mu\in\mathbb{C}, |\rho|<1.$

Kalla and Tuan [21] generalized Eq. (1.5) by means of the following integral and also obtained its asymptotic expansion

$$A_{\lambda,\gamma,\mu}^{(\alpha,\beta)}(\rho,\delta;k)=\int_0^{\pi}\frac{\cos^{2\alpha-1}(\theta/2)sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\left[1-\rho sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda}\left[1+\delta cos^2\left(\frac{\theta}{2}\right)\right]\mathrm{d}\theta, \tag{1.6}$$

where $0\leqslant k\leqslant 1, Re(\alpha)>0, Re(\beta)>0, \lambda,\mu,\gamma\in\mathbb{C}$ and either $|\rho|,|\delta|<1$ or ρ (or δ) $\in\mathbb{C}$ whenever $\lambda=m$ or $\gamma=-m$, $m\in\mathbb{N}_0$, respectively.

Al-Zamel et al. [2] discussed a generalized family of elliptic-type integrals in the form:

$$Z_{(\gamma)}^{(\alpha,\beta)}(k)=Z_{(\gamma_1,\cdots,\gamma_n)}^{(\alpha,\beta)}(k_1,\cdots,k_n)=\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n}\left(1-k_j^2\cos\theta\right)^{-\gamma_j}\mathrm{d}\theta,$$

$$=B(\alpha,\beta)\prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j}F_D^{(n)}\left(\beta;\gamma_1,\cdots,\gamma_n;\alpha+\beta;\frac{2k_1^2}{k_1^2-1},\cdots,\frac{2k_n^2}{k_n^2-1}\right), \tag{1.7}$$

where $Re(\alpha)>0, Re(\beta)>0, |k_j|<1; \gamma_j\in\mathbb{C}(j=1,\cdots,n), F_D^{(n)}$ is the Lauricella hypergeometric function of n variables [34].

Saxena and Kalla [29] have studied a family of elliptic- type integrals of the form :

$$\Omega_{(\sigma_1,\cdots,\sigma_{n-2};\delta,\mu)}^{(\alpha,\beta)}(\rho_1,\cdots,\rho_{n-2},\delta;k)=\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n-2}\left[1-\rho_j^2\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_j}$$

$$\left[1+\delta cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}[1-k^2\cos\theta]^{-\mu-\frac{1}{2}}\mathrm{d}\theta, \tag{1.8}$$

where $Re(\alpha)>0, Re(\beta)>0; \sigma_j(j=1,\cdots,n-2), \gamma,\mu\in\mathbb{C}; max\left\{|\rho_j|,\left|\frac{\delta}{1+\delta}\right|,\left|\frac{2k^2}{k^2-1}\right|\right\}<1.$

In the recent paper, Saxena and Pathan [26] investigated an extension of Eq.(1.8) in the form :

$$\Omega_{(\sigma_1,\cdots,\sigma_m,\gamma;\tau_1,\cdots,\tau_n)}^{(\alpha,\beta)}(\rho_1,\cdots,\rho_m,\delta;\lambda_1,\cdots,\lambda_n)=\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^{m}\left[1-\rho_i^2\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_i}$$

$$\left[1+\delta cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}\prod_{j=1}^{n}[1-\lambda_j^2\cos\theta]^{-\tau_j}\mathrm{d}\theta, \tag{1.9}$$

where $min(Re(\alpha), Re(\beta)) > 0; |\lambda_j| < 1; \sigma_i, \gamma, \tau_j \in \mathbb{C}; max\left\{|\rho_i|, \left|\frac{\delta}{1+\delta}\right|, \left|\frac{2\lambda_j^2}{\lambda_j^2-1}\right|\right\} < 1$ with

$(i = 1, \cdots, m; j = 1, \cdots, n)$.

In a recent paper [9], we have proposed and investigated a new family of unified and generalized elliptic-type integrals:

$$\bar{\Omega}_{\lambda_i,\tau_j}^{(\alpha,\beta)}((\rho_i),(\delta_i);k_j) = \bar{\Omega}_{\lambda_1,\cdots,\lambda_N,\tau_1,\cdots,\tau_M}^{(\alpha,\beta)}(\rho_1,\cdots,\rho_N,\delta_1,\cdots,\delta_N;k_1,\cdots,k_M) =$$

$$\int_0^{\pi} \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) sin^{2\beta-1}\left(\frac{\theta}{2}\right) \prod_{i=1}^{N}\left[1+\rho_i \sin^2\left(\frac{\theta}{2}\right)+\delta_i \cos^2\left(\frac{\theta}{2}\right)\right]^{-\lambda_j} \prod_{j=1}^{M}[1-k_j^2\cos\theta]^{-\tau_j}\mathrm{d}\theta, \tag{1.10}$$

where $min(Re(\alpha), Re(\beta)) > 0; |k_j| < 1; \lambda_i, \tau_j \in \mathbb{C}; max\left\{|\rho_i|, |\delta_i|, \left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|, \left|\frac{2k_j^2}{k_j^2-1}\right|\right\} < 1$ with

$(i = 1, \cdots, N; j = 1, \cdots, M)$.

which includes most of the known generalized and unified families of elliptic type integrals (including those discussed in (1.1) through (1.9)). For more details also see [17 ,27, 26, 1, 2, 24]. Upon a closer examination of the above equation. (1.10), it can be seen that the family of elliptic-type integral $\bar{\Omega}_{\lambda_i,\tau_j}^{(\alpha,\beta)}((\rho_i),(\delta_i);k_j)$ can be put in to the following form involving Euler-type integral:

$$\bar{\Omega}_{\lambda_1,\cdots,\lambda_N,\tau_1,\cdots,\tau_M}^{(\alpha,\beta)}(\rho_1,\cdots,\rho_N,\delta_1,\cdots,\delta_N;k_1,\cdots,k_M) = \prod_{j=1}^{M}(1-k_j^2)^{-\tau_j}\prod_{i=1}^{N}(1+\delta_i)^{-\lambda_i}$$

$$\int_0^1 \omega^{\beta}(1-\omega)^{\alpha-1}\prod_{j=1}^{M}\left[1-\left(\frac{2\omega k_j^2}{k_j^2-1}\right)\right]^{-\tau_j}\prod_{j=1}^{N}\left[1-\left(\frac{(\delta_i-\rho_i)\omega}{1+\delta_i}\right)\right]^{-\lambda_i}\mathrm{d}\omega, \tag{1.11}$$

A two-variables generating function $F(x,t)$ possess a formal power series representation in t, can be written in the following form

$$F(x,t) = \sum_{n=0}^{\infty} C_n f_n(x) t^n, \tag{1.12}$$

where each member of generalized set $\{f_n(x)\}_{n=0}^{\infty}$ is independant of t.

Special functions have been around for centuries. No one can imagine mathematics without Gaussian and confluent hypergeometric function, associated Legendre and Laguerre polynomials, Bessel functions and many more. The most well known application areas are in physics, engineering, chemistry, computer science and statistics. On several occasions, the solution of enumeration problems involving combinatorial objects requires knowledge from special function theory. Earlier the emphasis was on special functions satisfying linear differential equations, but this has now been extended to difference equations, partial differential equations, non linear differential equations and fractional differential equations[7,10].

The multivariable Aleph-function is an extension of the multivariable I-function recently defined by C.K. Sharma and Ahmad [31] , itself is a generalization of the multivariable H-function defined by Srivastava and Panda [37,38]. The multivariable Aleph-function is defined by means of the multiple contour integral :

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]$ $,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]$:
.... $,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]$:

$[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]$
$[(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\Bigg)$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\cdots,\xi_r)\prod_{k=1}^{r}\theta_k(\xi_k)z_k^{\xi_k}\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r \tag{1.13}$$

with $\omega=\sqrt{-1}$

$$\psi(\xi_1,\cdots,\xi_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}\xi_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}\xi_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}\xi_k)]} \tag{1.14}$$

and $$\theta_k(\xi_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}\xi_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}\xi_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}\xi_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}\xi_k)]} \tag{1.15}$$

For more details, see Ayant [3]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with}\ k=1,\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)} \tag{1.16}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.
We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,,min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$$

where $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k \tag{1.17}$$

For convenience, we will use the following notations in this paper.

$$V = m_1, n_1; \cdots ; m_r, n_r \tag{1.18}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots , p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.19}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,\mathfrak{n}}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}\} : \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\},$$

$$\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}; \cdots ; \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}} \tag{1.20}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{m+1,q_i}\}, \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}} ; \cdots ;$$

$$\{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}} \tag{1.21}$$

2. Main formulae

In this section we derive two new theorems and their corollaries on generating functions associated with multivariable ℵ -function and the families of elliptic-type integrals. These theorem and corollaries can be used to establish various known and new elliptic-type integrals. Some of the significant applications of the results derived in this section are discussed in the section 3. We have the general formula

Theorem 1

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} F[x, t\omega^{\eta}(1-\omega)^{\mu}] \aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V} \left(\begin{array}{c|c} (\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B} \end{array} \right) \mathrm{d}\omega = \Gamma(\gamma-\alpha) \sum_{n=0}^{\infty} C_n f_n(x) t^n (\gamma-\alpha)_{\mu n}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V} \left(\begin{array}{c|c} \mathrm{z}_1^{\xi_1} & (1\text{-}\alpha - n\eta; \xi_1, \cdots , \xi_r), A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma - n\eta - \mu\eta; \xi_1, \cdots , \xi_r), B \end{array} \right) \tag{2.1}$$

Provided that

$Re(\gamma - \alpha) > 0, Re(\eta) > 0, Re(\mu) > 0, \xi_i > 0$ for $i = 1, \cdots , r$

$Re(\alpha) + \sum_{i=1}^{r} \xi_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$ and $|arg z_k^{\xi_k}| < \frac{1}{2} A_i^{(k)} \pi$, $k = 1, \cdots , r$ where $A_i^{(k)}$ is given in (1.16).

Proof

Using the definition of two-variables generating function $F(x, t)$ in the series form with the help of (1.12), and the multivariable Aleph-function in Mellin-Barnes contour integral with the help of (1.13) and then interchanging the order of integration and summation suitably, which is permissible under the conditions stated above. . Collect the powers of ω and $(1 - \omega)$ and use the formula of Beta-integral. Interpreting the resulting Mellin-Barnes contour integral as an Aleph-function of r-variables, we arrive at the desired result.

Corollary 1

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}F[x,t\omega^{\eta}(1-\omega)^{\mu}]\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} (\mathrm{z}_1(1-\omega))^{\xi_1} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ (\mathrm{z}_r(1-\omega))^{\xi_r} & \mathrm{B} \end{array}\right)\mathrm{d}\omega = \Gamma(\alpha)$$

$$\sum_{n=0}^{\infty} C_n f_n(x) t^n (\gamma-\alpha)_{\mu n}\aleph^{0,\mathfrak{n}+1:V}_{p_i+1,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1^{\xi_1} & (1\text{-}\gamma+\alpha-n\mu;\xi_1,\cdots,\xi_r), A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma-n\eta-\mu\eta;\xi_1,\cdots,\xi_r), B \end{array}\right) \quad (2.2)$$

Provided that

$Re(\alpha)>0, Re(\eta)>0, Re(\mu)>0, \xi_i>0$ for $i=1,\cdots,r$

$Re(\gamma-\alpha)+\sum_{i=1}^{r}\xi_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ and $|argz_k^{\xi_k}|<\frac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.16).

Corollary 2

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}F[x,t\omega^{\eta}(1-\omega)^{\mu}]\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} (\mathrm{z}_1\omega(1-\omega))^{\xi_1} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ (\mathrm{z}_r\omega(1-\omega)^{\xi_r} & \mathrm{B} \end{array}\right)\mathrm{d}\omega = \Gamma(\gamma-\alpha)$$

$$\sum_{n=0}^{\infty} C_n f_n(x) t^n (\gamma-\alpha)_{\mu n}\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1^{\xi_1} & (1\text{-}\alpha-n\eta;\xi_1,\cdots,\xi_r),(1-\gamma+\alpha-n\mu;\xi_1,\cdots,\xi_r), A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma-n\eta-\mu\eta;2\xi_1,\cdots,2\xi_r), B \end{array}\right) \quad (2.3)$$

$Re(\eta)>0, Re(\mu)>0, \xi_i>0$ for $i=1,\cdots,r$

$Re(\alpha)+\sum_{i=1}^{r}\xi_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0, Re(\gamma-\alpha)+\sum_{i=1}^{r}\xi_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ and $|argz_k^{\xi_k}|<\frac{1}{2}A_i^{(k)}\pi$,

$k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.16).

The proofs of Corollary1 and 2 are similar to that of theorem 1.

Theorem 2

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} F[x,t\omega^{\eta}(1-\omega)^{\mu}]\prod_{l=1}^{M}\left[1-\left(\frac{2\omega k_j^2}{k_l^2-1}\right)\right]^{-\tau_l}\prod_{l'=1}^{N}\left[1-\left(\frac{(\delta_{l'}-\rho_{l'})\omega}{1+\delta_{l'}}\right)\right]^{-\lambda_{l'}}$$

$$\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} (\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B} \end{array}\right)\mathrm{d}\omega = \Gamma(\alpha)\sum_{n,\mathfrak{m}_l,\mathfrak{n}_{l'}=0}^{\infty} C_n f_n(x) t^n (\alpha)_{\mu n}$$

$$\frac{\prod_{l=1}^{M}(\tau_l)_{\mathfrak{m}_l}\prod_{l'=1}^{N}(\lambda_{l'})_{\mathfrak{n}_{l'}}}{\prod_{l=1}^{M}\mathfrak{m}_l!\prod_{l'=1}^{N}\mathfrak{n}_{l'}!}\prod_{j=1}^{M}\left[\left(\frac{2k_l^2}{k_l^2-1}\right)\right]^{\mathfrak{m}_l}\prod_{l'=1}^{N}\left[\left(\frac{\delta_{l'}-\rho_{l'}}{1+\delta_{l'}}\right)\right]^{\mathfrak{n}_{l'}}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c} z_1^{\xi_1} & (1\text{-}\beta-\sum_{l=1}^{M}\mathfrak{m}_l-\sum_{l'=1}^{N}\mathfrak{n}_{l'}-n\eta;\xi_1,\cdots,\xi_r),A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r^{\xi_r} & (1\text{-}\alpha-\beta-\sum_{l=1}^{M}\mathfrak{m}_l-\sum_{l'=1}^{N}\mathfrak{n}_{l'}-n\eta-\mu n;\xi_1,\cdots,\xi_r),B \end{array}\right) \tag{2.4}$$

where $\sum_{\mathfrak{m}_l,\mathfrak{n}_{l'}=0}^{\infty}=\sum_{\mathfrak{m}_1,\cdots,\mathfrak{m}_M,\mathfrak{n}_1,\cdots,\mathfrak{n}_N=0}^{\infty}$

provided that

$$max\left\{|\rho_{l'}|,|\delta_{l'}|,\left|\frac{\delta_{l'}-\rho_{l'}}{1+\delta_{l'}}\right|,\left|\frac{2k_l^2}{k_l^2-1}\right|\right\}<1 \text{ for } l=1,\cdots,M;l'=1,\cdots,N$$

$Re(\alpha)>0, Re(\eta)>0, Re(\mu)>0, \xi_i>0$ for $i=1,\cdots,r$

$\delta_{l'},\rho_{l'},\lambda_{l'},\tau_{l'}\in\mathbb{C};|k_l|<1$

$Re(\beta)+\sum_{i=1}^{r}\xi_i\min_{1\leqslant j\leqslant m_i}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ and $|argz_k^{\xi_k}|<\frac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.16).

Proof

To establish the integral formula (2.4), we first use the series representations for the generating function of two-variables $F(x,t)$ with the help of (1.12). Further, using contour integral representation for the multivariable Aleph-function with the help of (1.13) and then interchanging the order of integration and summation suitably, which is permissible under the conditions stated above. Then use the binomial expansion $M-$times (is valid)

$$\left[1-\left(\frac{2\omega k_j^2}{k_l^2-1}\right)\right]^{-\tau_l}=\sum_{\mathfrak{m}_l=0}^{\infty}\frac{(\tau_l)_{\mathfrak{m}_l}}{\mathfrak{m}_l!}\left(\frac{2\omega k_l^2}{k_l^2-1}\right)^{-\tau_l} \tag{2.5}$$

and the following binomial expansion $N-$times (is valid)

$$\prod_{l'=1}^{N}\left[1-\left(\frac{(\delta_{l'}-\rho_{l'})\omega}{1+\delta_{l'}}\right)\right]^{-\lambda_{l'}}=\sum_{\mathfrak{n}_{l'}=0}^{\infty}\frac{(\lambda_{l'})_{\mathfrak{n}_{l'}}}{\mathfrak{n}_{l'}!}\left[\left(\frac{(\delta_{l'}-\rho_{l'})\omega}{1+\delta_{l'}}\right)\right]^{\mathfrak{n}_{l'}} \tag{2.6}$$

Now, collect the powers of ω and $(1-\omega)$ and use the formula of Beta-integral and interpreting the resulting Mellin-Barnes contour integral as an Aleph-function of r-variables, we arrive at the desired result (2.4).

3. Applications

In view of the importance and usefulness of the theorems and corollaries discussed in the last section, we mention some interesting applications, which indicates manifold generality of the results obtained in this article.

(i) Consider the generating function [34]

$$F(x,t)=(1-tx)^{-\sigma}=\sum_{n=0}^{\infty}(\sigma)_n\frac{x^nt^n}{n!} \tag{3.1}$$

and use the theorem 1, under the state conditions, we obtain the following formulae.

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}[1-xt\omega^{\eta}(1-\omega)^{\mu}]^{-\sigma}\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}(\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\gamma-\alpha)$$

$$\sum_{n=0}^{\infty}\frac{(\sigma)_nx^nt^n(\gamma-\alpha)_{\mu n}}{n!}\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\alpha-n\eta;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma-n\eta-\mu\eta;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.2}$$

when we put $\omega=\cos^2\left(\frac{\theta}{2}\right)$ and $\cos\theta=2\cos^2\left(\frac{\theta}{2}\right)-1$ the above equation (3.2) gives the following generalization of the elliptic-type integral

$$\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)sin^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right)\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}\left[z_1\cos^2\left(\frac{\theta}{2}\right)\right]^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \left[z_r\cos^2\left(\frac{\theta}{2}\right)\right]^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\gamma-\alpha)$$

$$\sum_{n=0}^{\infty}\frac{(\sigma)_nx^nt^n(\gamma-\alpha)_{\mu n}}{n!}\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\alpha-n\eta;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma-n\eta-\mu\eta;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.3}$$

If we setting $\omega=\sin^2\left(\frac{\theta}{2}\right)$ and using $\cos\theta=1-2\sin^2\left(\frac{\theta}{2}\right)$ and $\sigma\to 0$ in (3.2), we have the following formula :

$$\int_0^{\pi}\sin^{2\alpha-1}\left(\frac{\theta}{2}\right)cos^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right)\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}\left[z_1\sin^2\left(\frac{\theta}{2}\right)\right]^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \left[z_r\sin^2\left(\frac{\theta}{2}\right)\right]^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\gamma-\alpha)$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\alpha;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.4}$$

It can be seen that the above elliptic-type integral (3.2) also provides generalization to a number of new families of elliptic-type integrals, which also generalizes known families of elliptic integrals. Also by using the generating function (3.1) and by the application of the theorem 2, under the stated conditions, we have obtained the following new family of elliptic-type integrals, which also generalizes known families of elliptic-type integrals.

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}\left[1-xt\omega^{\eta}(1-\omega)^{\mu}\right]^{-\sigma}\prod_{l=1}^{M}\left[1-\left(\frac{2\omega k_j^2}{k_l^2-1}\right)\right]^{-\tau_l}\prod_{l'=1}^{N}\left[1-\left(\frac{(\delta_{l'}-\rho_{l'})\omega}{1+\delta_{l'}}\right)\right]^{-\lambda_{l'}}$$

$$\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}(\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\alpha)\sum_{n,\mathfrak{m}_l,\mathfrak{n}_{l'}=0}^{\infty}\frac{(\sigma)_n x^n t^n(\alpha)_{\mu n}}{n!}$$

$$\frac{\prod_{l=1}^{M}(\tau_l)_{\mathfrak{m}_l}\prod_{l'=1}^{N}(\lambda_{l'})_{\mathfrak{n}_{l'}}}{\prod_{l=1}^{M}\mathfrak{m}_l!\prod_{l'=1}^{N}\mathfrak{n}_{l'}!}\prod_{j=1}^{M}\left[\left(\frac{2k_l^2}{k_l^2-1}\right)\right]^{\mathfrak{m}_l}\prod_{l'=1}^{N}\left[\left(\frac{\delta_{l'}-\rho_{l'}}{1+\delta_{l'}}\right)\right]^{\mathfrak{n}_{l'}}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'}-n\eta;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\alpha-\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'}-n\eta-\mu n,;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.5}$$

If we setting $\omega=\sin^2\left(\frac{\theta}{2}\right)$ and $\sigma\rightarrow 0$, we obtain

$$\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{l'=1}^{N}\left[1+\rho_{l'}\sin^2\left(\frac{\theta}{2}\right)+\delta_{l'}\cos^2\left(\frac{\theta}{2}\right)\right]^{-\lambda_{l'}}\prod_{l=1}^{M}[1-k_l^2\cos\theta]^{-\tau_l}$$

$$\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}\left[z_1\sin^2\left(\frac{\theta}{2}\right)\right]^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \left[z_r\sin^2\left(\frac{\theta}{2}\right)\right]^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\alpha)\prod_{l=1}^{M}(1-k_l^2)^{-\tau_l}\prod_{l'=1}^{N}(1+\delta_{l'})^{\lambda_{l'}}\sum_{\mathfrak{m}_l,\mathfrak{n}_{l'}=0}^{\infty}$$

$$\frac{\prod_{l=1}^{M}(\tau_l)_{\mathfrak{m}_l}\prod_{l'=1}^{N}(\lambda_{l'})_{\mathfrak{n}_{l'}}}{\prod_{l=1}^{M}\mathfrak{m}_l!\prod_{l'=1}^{N}\mathfrak{n}_{l'}!}\prod_{j=1}^{M}\left[\left(\frac{2k_l^2}{k_l^2-1}\right)\right]^{\mathfrak{m}_l}\prod_{l'=1}^{N}\left[\left(\frac{\delta_{l'}-\rho_{l'}}{1+\delta_{l'}}\right)\right]^{\mathfrak{n}_{l'}}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'};\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\alpha-\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'};\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.6}$$

(ii) Consider the following generating function [34]

$$F(x,t)=(1-x_1t)^{-\sigma_1}(1-x_2t)^{-\sigma_2}=\sum_{n=0}^{\infty}g_n^{\sigma_1,\sigma_2}(x_1,x_2)t^n \tag{3.7}$$

where

$$g_n^{\sigma_1,\sigma_2}(x,y)=\sum_{r=0}^{n}\frac{(\sigma_1)_r(\sigma_2)_{n-r}}{r!(n-r)!}x^r y^{n-r} \tag{3.8}$$

and by the application of the theorem 1 under the state conditions, we obtain

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}(1-x_1t)^{-\sigma_1}(1-x_2t)^{-\sigma_2}\,\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}(\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\gamma-\alpha)$$

$$\sum_{n=0}^{\infty} g_n^{\sigma_1,\sigma_2}(x_1,x_2)t^n\,(\gamma-\alpha)_{\mu n}\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\alpha-n\eta;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma-n\eta-\mu\eta;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.9}$$

and by the application of the theorem 2 under the state conditions, we obtain

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}\,(1-x_1t)^{-\sigma_1}(1-x_2t)^{-\sigma_2}\prod_{l=1}^{M}\left[1-\left(\frac{2\omega k_j^2}{k_l^2-1}\right)\right]^{-\tau_l}\prod_{l'=1}^{N}\left[1-\left(\frac{(\delta_{l'}-\rho_{l'})\omega}{1+\delta_{l'}}\right)\right]^{-\lambda_{l'}}$$

$$\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}(\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\alpha)\sum_{n,\mathfrak{m}_l,\mathfrak{n}_{l'}=0}^{\infty} g_n^{\sigma_1,\sigma_2}(x_1,x_2)t^n(\alpha)_{\mu n}$$

$$\frac{\prod_{l=1}^{M}(\tau_l)_{\mathfrak{m}_l}\prod_{l'=1}^{N}(\lambda_{l'})_{\mathfrak{n}_{l'}}}{\prod_{l=1}^{M}\mathfrak{m}_l!\prod_{l'=1}^{N}\mathfrak{n}_{l'}!}\prod_{j=1}^{M}\left[\left(\frac{2k_l^2}{k_l^2-1}\right)\right]^{\mathfrak{m}_l}\prod_{l'=1}^{N}\left[\left(\frac{\delta_{l'}-\rho_{l'}}{1+\delta_{l'}}\right)\right]^{\mathfrak{n}_{l'}}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'}-n\eta;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\alpha-\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'}-n\eta-\mu n,;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.10}$$

(iii) Consider the following generating function

$$F(x,t)=e^{-xt}=\sum_{n=0}^{\infty}\frac{(-)^n x^n t^n}{n!} \tag{3.11}$$

and by the application of the theorem 1 under the state conditions, we obtain

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}e^{-xt[\omega^{\eta}(1-\omega)^{\mu}]}\,\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}(\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\gamma-\alpha)$$

$$\sum_{n=0}^{\infty}\frac{(-)^n(\gamma-\alpha)_{\mu n}x^nt^n}{n!}\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\alpha-n\eta;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\gamma-n\eta-\mu\eta;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.12}$$

and by the application of the theorem 2 under the state conditions, we obtain

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}\,e^{-xt[\omega^\eta(1-\omega)^\mu]}\prod_{l=1}^{M}\left[1-\left(\frac{2\omega k_j^2}{k_l^2-1}\right)\right]^{-\tau_l}\prod_{l'=1}^{N}\left[1-\left(\frac{(\delta_{l'}-\rho_{l'})\omega}{1+\delta_{l'}}\right)\right]^{-\lambda_{l'}}$$

$$\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}(\mathrm{z}_1\omega)^{\xi_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ (\mathrm{z}_r\omega)^{\xi_r} & \mathrm{B}\end{array}\right)\mathrm{d}\omega=\Gamma(\alpha)\sum_{n,\mathfrak{m}_l,\mathfrak{n}_{l'}=0}^{\infty}\frac{(-)^n(\alpha)_{\mu n}x^nt^n}{n!}$$

$$\frac{\prod_{l=1}^{M}(\tau_l)_{\mathfrak{m}_l}\prod_{l'=1}^{N}(\lambda_{l'})_{\mathfrak{n}_{l'}}}{\prod_{l=1}^{M}\mathfrak{m}_l!\prod_{l'=1}^{N}\mathfrak{n}_{l'}!}\prod_{j=1}^{M}\left[\left(\frac{2k_l^2}{k_l^2-1}\right)\right]^{\mathfrak{m}_l}\prod_{l'=1}^{N}\left[\left(\frac{\delta_{l'}-\rho_{l'}}{1+\delta_{l'}}\right)\right]^{\mathfrak{n}_{l'}}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1^{\xi_1} & (1\text{-}\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'}-n\eta;\xi_1,\cdots,\xi_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r^{\xi_r} & (1\text{-}\alpha-\beta-\sum_{l=1}^{M}\mathrm{m}_l-\sum_{l'=1}^{N}\mathrm{n}_{l'}-n\eta-\mu n,;\xi_1,\cdots,\xi_r),B\end{array}\right) \tag{3.13}$$

Remarks

We obtain the similar formulae concerning the multivariable I-function [31], the multivariable H-function [37,38], the Aleph-function of two variables [30] and the I-function of two variables [32].

If the multivariable Aleph-function reduces to Aleph-function of one variable [39,40], we obtain the recently results of Chaurasia and Gill [6], this work is a generalization of the results given by Chaurasia and Singh [11] and Chaurasia and Meghwal [9].

4. Conclusion

In this paper, we have presented a solution of generalized elliptic type integral with multivariable Aleph-function. The solution has been developed in a compact and elegant form with the help of generating functions, multivariable Aleph -function is general in nature and includes a number of known and new results as particular cases. This extended elliptic type integral used to compute the certain problems of radiation physics, nuclear technology and may be utilized in other branch of mathematics.

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