INTERSECTION CORDIAL LABELING OF GRAPHS

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Abstract **- An intersection cordial labeling of a graph G with vertex set V is an injection f from V to the power set of $\{1, 2, \ldots, n\}$ such that if each edge uv is assigned the label 1 if $f(u) \cap f(v) \neq \emptyset$ and 0 otherwise; Then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has an intersection cordial labeling, then it is called intersection cordial graph. In this paper, we proved the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are intersection cordial. We also proved that complete graph is not intersection cordial.**

Key words: Cordial labeling, Intersection cordial labeling, Intersection cordial graphs.

1 INTRODUCTION

By a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [5]. Here G denotes $(p, q)-$ graph, where p is the number of vertices and q is the number of edges of G.

First we give the set concepts in Algebra [3].

Let $X = \{1, 2, \ldots, n\}$ be a set and $\wp(X)$ be the collection of all subsets of X, called power set of X.

If A is a subset of B, we denote it by $A \subset B$ otherwise by $A \not\subset B$. Note that $\wp(X)$ contains 2^n subsets. Let nCr denotes the number of ways of selecting r objects from n objects.

Graph labeling [4] is a strong communication between Algebra [3] and structure of graphs [5]. By combining the set theory concept in Algebra and Cordial labeling concept in Graph labeling, we introduce a new concept called intersection cordial labeling.

A vertex labeling [4] of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a label depending on the vertex label $f(u)$ and $f(v)$. The two best known labeling methods are called graceful and harmonious labeling. Cordial labeling is a variation of both graceful and harmonious labeling [1].

Definition 1.1. *[1] Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$ and $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0)$ and $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

The concept of cordial labeling was introduced by Cahit [1].

Definition 1.2. *[2] A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.*

Cahit proved some results in [2].

2 Main Results

D.K Nathan and K.Nagarajan have introduced subset cordial labeling and they have proved that some graphs are subset cordial [6].

Definition 2.1. *[6] Let $X = \{1, 2, \ldots, n\}$ be a set. Let $G = (V, E)$ be a simple $(p, q)-$ graph and $f : V \to \wp(X)$ be an injection. Also, let $2^{n-1} < p \leq 2^n$. For each edge uv, assign label 1 if either $f(u) \subset f(v)$ or $f(v) \subset f(u)$ or assign 0 if $f(u)$ is not a subset of $f(v)$ and $f(v)$ is not a subset of $f(u)$. f is called a subset cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.*

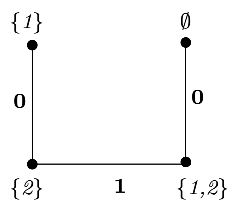
A graph is called a subset cordial graph if it has a subset cordial labeling.

Motivated by the concept of subset cordial labeling, we introduce a new special type of cordial labeling called intersection cordial labeling as follows.

Definition 2.2. *Let $X = \{1, 2, \ldots, n\}$ be a set. Let $G = (V, E)$ be a simple $(p, q)-$ graph and $f : V \to \wp(X)$ be an injection. Also, let $2^{n-1} < p \leq 2^n$. For each edge uv, assign label 1 if either $f(u) \cap f(v) \neq \emptyset$ and 0 if $f(u) \cap f(v) = \emptyset$. f is called a intersection cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.*

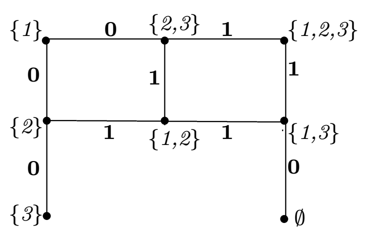
A graph is called an intersection cordial graph if it has an intersection cordial labeling.

Example 2.3. *Consider the following graph G. Take $X = \{1, 2\}$.*



Here $e_f(0) = 2$ and $e_f(1) = 1$, $|e_f(0) - e_f(1)| = 1$. Thus G is intersection cordial.

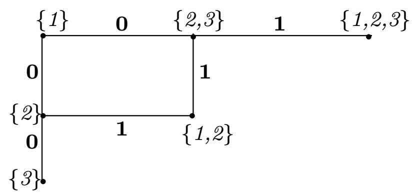
Example 2.4. *Consider the following graph G. Take $X = \{1, 2, 3\}$.*



Here $e_f(0) = 4$ and $e_f(1) = 5$, $|e_f(0) - e_f(1)| = 1$.
Thus G is intersection cordial.

Remark 2.5. *If $2^{n-1} < p < 2^n$, we take $X = \{1, 2, \ldots, n\}$, then we have more number of subsets than vertices. So we can easily label the vertices so that $|e_f(0) - e_f(1)| \leq 1$. So, for proving intersection cordiality of graph, it enough to prove for $p = 2^n$.*

Example 2.6. *In the example 2.4, by removing the vertices labeled $\{1, 3\}$ and $\emptyset$, we get the following intersection cordial graph.*



In this paper, we prove the standard graphs such as path, cycle, star, wheel and some complete bipartite graphs are intersection cordial. We also prove that complete graph is not an intersection cordial.

First we prove that star graphs are intersection cordial.

Theorem 2.7. *The star graph $K_{1,q}$ is intersection cordial.*

Proof. Let $X = \{1, 2, \ldots, n\}$.

Assume that $p = 2^n$.

Let v_p be the central vertex and $v_1, v_2, \ldots, v_{p-1}$ be the end vertices of the star.

Note that, $K_{1,q}$ containing $q = 2^n - 1$ edges. First, we label the central vertex by any 1-element set, say $\{1\}$. Label the remaining end vertices by the other subsets of X.

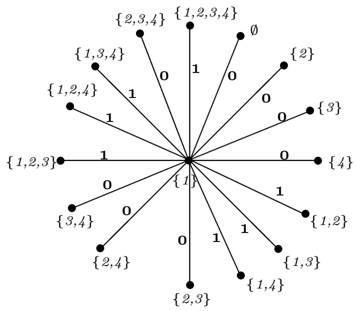
We observe that the intersection of each subset of $\{2, 3, 4, \ldots, n\}$ with $\{1\}$ is $\emptyset$ and so it contributes 2^{n-1}, 0's to $e_f(0)$.

The remaining subsets contribute $2^{n-1} - 1$, 1's to $e_f(1)$.

Thus $|e_f(0) - e_f(1)| = 1$ and so the star graph $K_{1,q}$ is intersection cordial. □

The labeling pattern given in the Theorem 2.7 is illustrated in the following example.

Example 2.8. *Consider the following star graph G. Take $X = \{1, 2, 3, 4\}$*



Here $e_f(0) = 8$ *and* $e_f(1) = 7$

Thus $|e_f(0) - e_f(1)| = 1$ *and hence* G *is intersection cordial.*

Next, the intersection of cordiality of complete graphs are discussed. It is clear that K_p is intersection cordial for $p = 1, 2$ and 3. We will prove that K_p is not intersection cordial for $p \geq 4$.

Theorem 2.9. *K_{2^n} is not intersection cordial for $n \geq 2$.*

Proof. Let $X = \{1, 2, 3, \ldots, n\}$ and $p = 2^n$.

Label the vertices of K_{2^n} by all subsets of X.

We first calculate $e_f(0)$. Note that the intersection of the empty set $\emptyset$ with any subset of X is $\emptyset$ and it contributes $2^n - 1$ to $e_f(0)$.

Now consider the singleton sets. For example, take $\{1\}$.

The number of subsets of the set $\{2, 3 \ldots n\}$ is 2^{n-1}. We see that the intersection of the subsets of $\{1\}$ with any subset of $\{2, 3, , \ldots, n\}$ is $\emptyset$ and the empty subset has already be taken in account. So, it contributes $2^{n-1} - 1$ to $e_f(0)$. Since the number of singleton sets is n, all the singleton sets contribute $n2^{n-1}$ to $e_f(0)$ and so $e_f(0) = \binom{n}{1}2^{n-1}$.
Next, we consider the two elements sets. For example, take $\{1, 2\}$. The number of subsets of the set $\{3, 4, \ldots, n\}$ is 2^{n-2}. The intersection of the set $\{1, 2\}$ with every subset of $\{3, 4, \ldots, n\}$ is $\emptyset$ and so it contributes 2^{n-2} to $e_f(0)$. There are $\binom{n}{1}$ two elements sets, and so these two elements subsets contribute $\binom{n}{2}2^{n-2}$ to $e_f(0)$.

Similarly, $\binom{n}{3}$ three elements subsets contribute $\binom{n}{3}2^{n-3}$ to $e_f(0)$. In general, all the k-members subsets contribute $\binom{n}{k}2^{n-k}$ to $e_f(0)$.

In a complete graph, we observe that each contribution counted as twice.

Hence

$$\begin{aligned}
e_f(0) &= \frac{1}{2}[2^n - 1 + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{k}2^{n-k} + \binom{n}{n}] \\
&= \frac{1}{2}[2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{n} - 1] \\
&= \frac{1}{2}[(2+1)^n - 1] \\
&= \frac{3^n - 1}{2}.
\end{aligned}$$

We have $e_f(0) + e_f(1) = q = \frac{p(p-1)}{2}$.
Then

$$\begin{aligned}
e_f(1) &= \frac{p(p-1)}{2} - e_f(0) \\
&= \frac{2^n(2^n - 1)}{2} - \frac{3^n - 1}{2} \\
&= 2^{n-1}(2^n - 1) - \frac{3^n - 1}{2}.
\end{aligned}$$

Now

$$\begin{aligned}
| e_f(0) - e_f(1) | &= | \frac{3^n - 1}{2} - 2^{n-1}(2^n - 1) + \frac{3^n - 1}{2} | \\
&= (3^n - 1) - 2^{n-1}(2^n - 1) \geq 2 \text{ for } n \geq 2.
\end{aligned}$$

Thus K_{2^n} is not intersection cordial. □

Next, we prove that path is intersection cordial.

Theorem 2.10. *The path P_{2^n} is intersection cordial.*

Proof. Let $X = \{1, 2, ..., n\}$ and let $p = 2^n$
.

Then $q = 2^n - 1$.

Let $V(P_{2^n}) = \{v_1, v_2, ...v_{2^n}\}$.

Now, we arrange all the subsets of X in the following pattern. Label the first vertex of P_{2^n} by $\emptyset$ and the second vertex of P_{2^n} by X.

Next, label the third vertex of P_{2^n} by 1-element set and fourth vertex by its complement. Continue this process until, the 1-element subsets are exhausted. After that we label the 2-element subset and that next by its complement. Continue this process, until the 2-element subsets are exhausted.

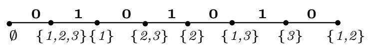
Now, continuing by labels of 3-element, 4-element subsets and this process will end upto $\frac{n}{2}$ element subset if n is even. The k-element subsets were $(k > \frac{n}{2})$ are already labeled and so the labeling is completed. Thus all the subsets of X are exhausted.

If n is odd, we continue the above process upto $\frac{n-1}{2}$ - element subsets.

Now, we see the labeling of edges of P_{2^n}. Clearly the labeling of first edge in P_{2^n} is 0. Note that the intersection of X and the 1-element set is non empty, so the labeling of second edge is 1. Next we labeled the vertices by the set and its complement alternatively and so the labels 0 and 1's are labeled alternatively to the edges of P_{2^n} .

Thus we have $e_f(0) = 2^{n-1}$ and $e_f(1) = 2^{n-1} - 1$ and so $|e_f(0) - e_f(1)| = 1$ Hence P_{2^n} is intersection cordial. □

Example 2.11. *Consider the path* $P_{2^3} = P_8$.



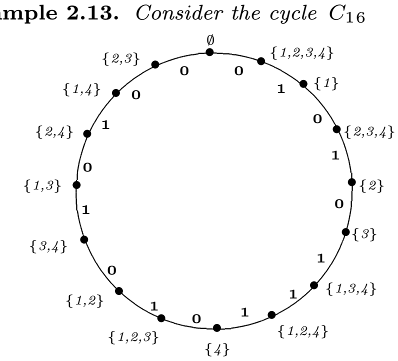
Here $e_f(0) = 4; e_f(1) = 3$. *Thus* $| e_f(0) - e_f(1) |= 1$.

Theorem 2.12. *The cycle* C_{2^n} *is intersection cordial.*

Proof. Consider the labeling of path P_{2^n} in the Theorem 2.10 . Now joining the initial and finial vertices of P_{2^n} we get the cycle C_{2^n} and the new edge joining initial and finial vertices of path get the label 0 .

We see that $e_f(0) = 2^{n-1} + 1$ and $e_f(1) = 2^{n-1} - 1$. But $|e_f(0) - e_f(1)| = 2$. To make intersection cordiality, we interchange the labeling of vertices already labeled by $\{1, 2, ..., n\}$ and $\{1\}$. Now we get $e_f(0) = e_f(1) = 2^{n-1}$ and $|e_f(0) - e_f(1)| = 0$. So C_{2^n} is intersection cordial. □

Example 2.13. *Consider the cycle* C_{16}



Here $e_f(0) = 8, e_f(1) = 8$. *Thus* $| e_f(0) - e_f(1) |= 0$.

Now, we will prove some complete bipartite graphs are intersection cordial.

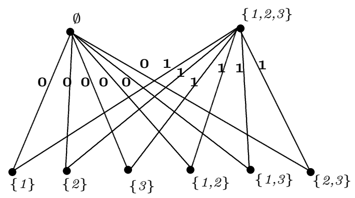
Theorem 2.14. $K_{2,2^n-2}$ *is intersection cordial.*

Proof. Let $V(K_{2,2^n-2}) = A \cup B$, where $A = \{u_1, u_2\}$ and $B = \{v_1, v_2, ..., v_{2^n-2}\}$. Consider the set $X = \{1, 2, ..., n\}$. Now we label all the subsets of X to the vertices of $K_{2,2^n-2}$ as follows. First, we label the vertex u_1 by $\emptyset$ and u_2 by the whole set X . Next we label the remaining subsets of X to the vertices $v_1, v_2, ..., v_{2^n-2}$.

The labeling of edges incident with u_1 are 0 and incident with u_2 are 1, since the intersection of $\emptyset$ with any set is $\emptyset$ and intersection of whole set X with any subset Y of X is Y .

Thus , we see that $e_f(0) = 2^n - 2 = e_f(1)$.
Hence $K_{2,2^n-2}$ is intersection cordial. □

Example 2.15. *Consider the graph* $K_{2,6}$



Here $e_f(0) = 6$ *and* $e_f(1) = 6$.
Thus the complete bipartite graph $K_{2,6}$ *is intersection cordial.*

Theorem 2.16. $K_{3,2^n-3}$ *is intersection cordial.*

Proof. Note that $K_{3,2^n-3}$ has 2^n vertices and $3.2^n - 9$ edges.
Let $V(K_{3,2^n-3}) = A \cup B$ where $A = \{u_1, u_2, u_3\}$ and $B = \{v_1, v_2, ..., v_{2^n-3}\}$. Consider the set $X = \{1, 2, 3, ..., n\}$. Now we label all the subsets of X to the vertices $K_{3,2^n-3}$ as follows.

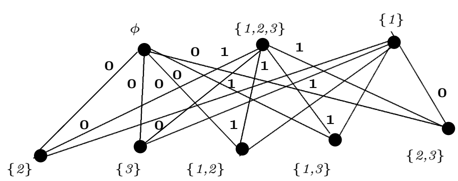
First we label the vertex u_1 by $\emptyset$, u_2 by the whole set X and u_3 by any 1-element subset, say $\{1\}$. Next we label the remaining subsets of X to the vertices $v_1, v_2, ..., v_{2^n-3}$.

The labeling of edges incident with u_1, u_2 and u_3 are given in the following table.

No.	Edges incident with	$e_f(0)$	$e_f(1)$
i)	u_1	$2^n - 3$	0
ii)	u_2	0	$2^n - 3$
iii)	u_3	$2^{n-1} - 1$	$2^{n-1} - 2$
	Total	$3.2^{n-1} - 4$	$3.2^{n-1} - 5$

Thus $| e_f(0) - e_f(1) |= 1$ and so $K_{3,2^n-3}$ is intersection cordial. □

Example 2.17. *Consider the following complete bipartite graph* $K_{3,5}$



Here $e_f(0) = 8$ *and* $e_f(1) = 7$, *and* $| e_f(0) - e_f(1) |= 1$.
Thus, the complete bipartite graph $K_{3,5}$ *is intersection cordial.*

Conclusion 2.18. *In this paper, we have checked intersection cordiality of some standard graphs. Further, we are trying to establish the intersection cordiality of the following various structure of graphs.*
(i) Subdivision of standard graphs
(ii) Wheel related graphs
(iii) Cycle related graphs
(iv) Star related graphs
(v) Corona of graphs
(vi) Transformation graphs

Also, we are going to establish the relationship between intersection cordiality and other cordiality such as subset cordiality, divisor cordiality, prime cordiality etc.

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