gc - domination and GC-domination numbers of a graph

A.Muthukamatchi
Department of Mathematics, Government Arts College for women, Nilakkottai-624 208
Tamil Nadu, India

Abstract: Throughout this paper, we assume that G = (V,E) is a finite, simple connected graph with at least two vertices. Acharya and Sampathkumar [2] introduced the concept of graphoidal covers and graphoidal covering number of a graph. Arumugam and Suresh Suseela [4] introduced the concept of acyclic graphoidal cover and acyclic graphoidal covering number of a graph. An elaborate review of results in graphoidal covers with several interesting applications and a collection of unsolved problems is given in [3]. Any graph theoretic concept which depends only on adjacency of vertices can be extended in the context of graphoidally covered graph and ψ = E(G) yields the original concept as a special case.

A *graphoidal cover* of a graph G is a collection of paths(not necessarily open) in G satisfying the following conditions.

(i) Every path in ψ has at least two vertices.

(ii) Every vertex of G in an internal vertex of at most one path in ψ.

(iii) Every edge of G in exactly one path in ψ.

A graphoidal cover ψ of a graph G is called an *acyclic graphoidal cover* if every member of ψ is a path. we assume throughout that ψ is an acyclic graphoidal cover of G.

Let G be a connected graph. Given an acyclic graphoidal cover ψ of G, we associate with the pair (G,ψ) another graph with vertex set V (G) which we denote by G(ψ).

Given any graph theoretic parameter Ω, we can use the concept of acyclic graphoidal covers to define two new parameters associated with Ω as follows:

For any acyclic graphoidal cover ψ of G, let $\Omega_\psi = \Omega(G(\psi))$. We now define $\Omega_{gc}(G) = \min\{\Omega(G(\psi))\}$ and $\Omega_{GC}(G) = \max\{\Omega(G(\psi))\}$, where the minimum and maximum are taken over all acyclic graphoidal covers ψ of G. Since $\Omega_\psi(G) = \Omega(G)$, where ψ = E(G), we have $\Omega_{gc}(G) \le \Omega(G) \le \Omega_{GC}(G)$. We now proceed to study $\Omega_{gc}(G)$ and $\Omega_{GC}(G)$, where Ω is a domination related parameter.

We first determine $\gamma_{gc}(G)$ and $\gamma_{GC}(G)$ for standard graphs.

The parameters $\gamma_{gc}(G)$ and $\gamma_{GC}(G)$ are respectively called the gc-domination number and GC-domination number of G.

Theorem 1.1. *Let G be any graph. Then $\gamma(G) = 1$ if and only if $\gamma_{gc}(G) = 1$.*

Proof. Since $\gamma_{gc}(G) \le \gamma(G)$, it follows that if $\gamma(G) = 1$, then $\gamma_{gc}(G) = 1$.

Conversely, let $\gamma_{gc}(G) = 1$. Let ψ be an acyclic graphoidal cover of G such that $\gamma_{gc}(G) = \gamma_\psi(G) = 1$. Then there exists a vertex, say v_1, such that v_1 is ψ-adjacent to all the vertices of $V - \{v_1\}$. Let P_i be the v_1 - v_i path in ψ, $2 \le i \le n$. If P_i has length greater than 1 for some i, then v_1 is not ψ-adjacent to the vertex w_1 which is adjacent to v_i and is on P_i. Hence each P_i has length 1, so that $\deg_G(v_1) = n - 1$. Thus $\gamma(G) = 1$.

Corollary 1.2. If $\gamma(G) = 2$, then $\gamma_{gc}(G) = 2$.

Proof. Since $\gamma_{gc}(G) \leq \gamma(G) = 2$ and $\gamma_{gc}(G) \neq 1$, we have $\gamma_{gc}(G) = 2$.

Remark 1.3. *The converse of Corollary 1.2 is not true.*

Example 1.4. Consider the acyclic graphoidal cover $\psi = \{(1,2),(2,3),(3,4),(2,5),(3,5,6)\}$ of the graph G given in Figure 1.1(a). Then $G(\psi)$ is given in Figure 1.1(b)

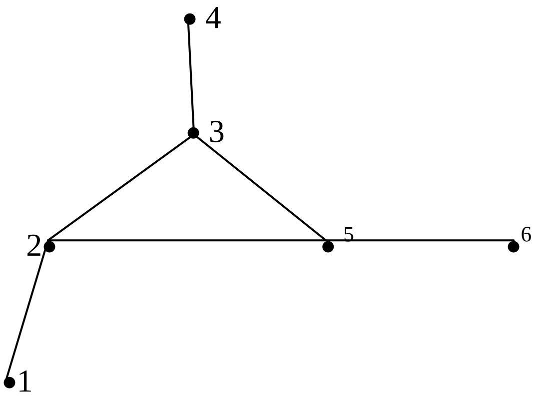

Figure 1.1 (a)

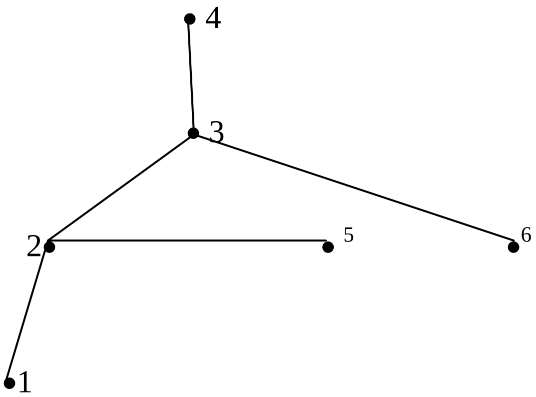

Figure 1.1(b)

Here $\gamma(G) = 3$ and $\gamma_{gc}(G) = 2$.

Theorem 1.5.*For the star $K_{1,n}$, $\gamma_{gc}(K_{1,n}) = 1$ and $\gamma_{GC}(K_{1,n}) = 2$.*

Proof. Let V (G) = $\{v_0,v_1,v_2,\ldots,v_n\}$ with deg v_0 = n and deg $v_i = 1$ for all $i = 1,2,\ldots,n$.

Let ψ be any acyclic graphoidal cover of G. If v_0 is interior to ψ, then $G(\psi)$ is isomorphic to $K_2 \cup K_{1,\, n-2}$ and hence $\gamma_\psi(G) = 2$.

If v_0 is exterior to ψ, then $\psi = E(G)$ and hence $\gamma_\psi(G) = 1$.

Thus for any acyclic graphoidal cover ψ, we have $\gamma_\psi(G) = 1$ or 2 and hence $\gamma_{gc}(G) = 1$ and $\gamma_{gc}(G) = 2$.

Theorem 1.6. *For the bistar $G=B(n_1,n_2)$ where $n_1,n_2 \geq 3$, we have $\gamma_{gc}(B(n_1,n_2)) = 2$ and $\gamma_{GC}(B(n_1,n_2)) = 4$.*

Proof.

Let V $(B(n_1,n_2)) = \{u,v,u_1,u_2,\ldots,u_{n_1},v_1,v_2,\ldots,v_{n_2}\}$ with $deg\ u_i = deg\ v_j = 1$ for all $1 \leq i \leq n_1, 1 \leq j \leq n_2$, $N[u] = \{u,u_1,u_2,\ldots,u_{n_1}\}$ and $N[v] = \{v,v_1,v_2,\ldots,v_{n_2}\}$.

Let ψ be any acyclic graphoidal cover of $B(n_1,n_2)$. Let S be the set of vertices which are interior to ψ.

Case i. $|S| = 0$.

Then $\psi = E(G)$ and hence $\gamma_\psi(B(n_1,n_2)) = 2$.

Case ii. $|S| = 1$.

Assume without loss of generality that $S = \{u\}$. Let P be the path in ψ having u as an internal vertex. Then $P = (u_i,u,u_j)$ for some i , j where $1 \leq i < j \leq n_1$ or $P = (u_i,u,v)$ for some i,$1 \leq i \leq n_1$. Hence $\psi = \{P\} \cup \{E(G) \setminus E(P)\}$ so that $G(\psi)$ is isomorphic to $K_2 \cup B(n_1 - 2,n_2)$ or $K_{1,n_2+1} \cup K_{1,n_1-1}$. Thus $\gamma_\psi(B(n_1,n_2)) = 3$ or 2.

Case iii. Let $|S| = 2$.

Then $S = \{u,v\}$. If both u and v are internal vertices of the same path P in ψ then $P = (u_i,u,v,v_j)$ for some i, j where $1 \leq i \leq n_1, 1 \leq j \leq n_2$ and $\psi = \{P\} \cup \{E(G) \setminus E(P)\}$. In this case $G(\psi)$ is isomorphic to $K_2 \cup K_{1,n_1-2} \cup K_{1,n_2-2}$ and hence $\gamma_\psi(B(n_1,n_2)) = 3$.

If u and v are internal vertices of two different paths P_1 and P_2 in ψ, then $P_1 = (u_i,u,u_j)$ for some i, j, $1 \leq i < j \leq n_1$, and $P_2 = (v_r,v,v_s)$ for some r, s, $1 \leq r < s \leq n_2$, and $\psi = \{P_1,P_2\} \cup \{E(G) \setminus E(P_1 \cup P_2)\}$.

In this case $G(\psi)$ is isomorphic to $2K_2 \cup B(n_1 - 2 , n_2 - 2)$ and hence $\gamma_\psi(B(n_1,n_2)) = 4$.

Thus $\gamma_{gc}(B(n_1,n_2)) = 2$ and $\gamma_{GC}(B(n_1,n_2)) = 4$.

Theorem 1.7. *For the tree $T = S(K_{1,n})$, we have $\gamma_{gc}(T) = n$ and $\gamma_{GC}(T) = n + 2$.*

Proof. Let V (T) = $\{u, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and E(T) = $\{uu_i, u_iv_i : i = 1,2,\ldots,n\}$. Let ψ be any acyclic graphoidal cover of T. Let S be the set of all vertices which are interior to ψ.

Case i. S = φ.

Then ψ = E(T) and $\gamma_\psi(T) = \gamma(T) = n$.

Case ii. S = {u}.

Without loss of generality we can take ψ = $\{(u_1,u,u_2),(u_1,v_1),(u_2,v_2),\ldots,(u_n,v_n),(u,u_3),(u,u_4),\ldots,(u,u_n)\}$. In this case T(ψ) is isomorphic to $P_4 \cup S(K_{1,n-2})$ and hence $\gamma_\psi(T) = n$.

Case iii. S = $\{u_1,u_2,\ldots,u_n\}$.

In this case ψ = $\{uu_iv_i : i = 1,2,\ldots,n\}$ and T(ψ) is isomorphic to $nK_1 \cup K_{1,n}$ and hence $\gamma_\psi(T) = n + 1$.

Case iv. S = $\{u,u_1,u_2,\ldots,u_n\}$.

In this case

ψ = $\{(v_1,u_1,u,u_2,v_2),(u,u_i,v_i) : i = 3,4,\ldots,n\}$ and T(ψ) is isomorphic to $nK_1 \cup K_2 \cup K_{1,n-2}$ and $\gamma_\psi(T) = n + 2$.

Case v. S = $\{u_1,u_2,\ldots,u_k\}$ for some k, where $1 \le k < n$.

Then ψ = $\{(v_1,u_1,u),(v_2,u_2,u),\ldots,(v_k,u_k,u),(u,u_{k+1}),(u_{k+1},v_{k+1}),\ldots,(u,u_n),(u_n,v_n)\}$. In this case T(ψ) is isomorphic to the graph given in Figure 1.2 and $\gamma_\psi(T) = n + 1$.

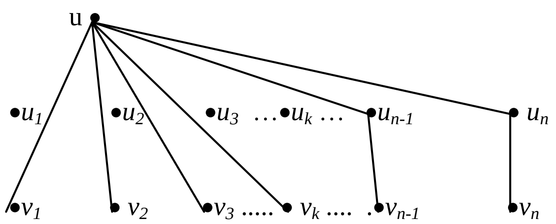

Figure 1.2

Case vi. S = $\{u,u_1,u_2,\ldots,u_k\}$ for some k, $1 \le k < n$. Then ψ = $\{(v_1,u_1,u,u_k,v_k),(v_2,u_2,u),\ldots,(v_{k-1},v_{k-1},u),(u,u_{k+1}),(u_{k+1},v_{k+1}),\ldots,(u,u_n),(u_n,v_n)\}$. In this case T(ψ) is isomorphic to the graph given in Figure 1.3 and $\gamma_\psi(T) = n + 2$.

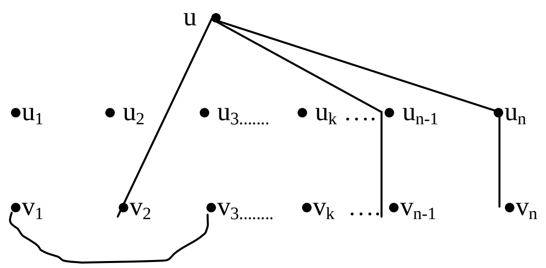

Figure 1.3

Hence $\gamma_{gc}(T) = n$ and $\gamma_{GC}(T) = n + 2$.

Theorem 1.8. For any path P_n, $\gamma_{gc}(P_n) = \lceil \frac{n}{3} \rceil$ and $\gamma_{GC}(P_n) = n - 1$.

Proof. Let ψ be any acyclic graphoidal cover of P_n.

Case i. $|\psi| = n - 1$.

Then ψ = $E(P_n)$ and $\gamma_\psi(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

Case ii. $|\psi| = 1$.

Then ψ = $\{P_n\}$ and $P_n(\psi)$ is isomorphic to $K_2 \cup (n-2)K_1$. Hence $\gamma_\psi(P_n) = n - 1$.

Case iii. Let ψ = $\{Q_1,Q_2,Q_3,\ldots,Q_r\}$, where $2 \le r \le n - 2$.

Then $P_n(\psi)$ is isomorphic to $P_{r+1} \cup (n - r - 1)K_1$.

Thus $\gamma_\psi(P_n) = \lceil \frac{r+1}{3} \rceil + (n - r - 1) \le n-1$.

Hence $\gamma_{gc}(P_n) = \lceil \frac{n}{3} \rceil$ and $\gamma_{GC}(P_n) = n - 1$.

Theorem 1.9. For any cycle C_n, $\gamma_{gc}(C_n) = \lceil \frac{n}{3} \rceil$ and $\gamma_{GC}(C_n) = n - 1$.

Proof. If ψ is any acyclic graphoidal cover of C_n with $|\psi| = 2$, then $C_n(\psi) = K_2 \cup (n - 2)K_1$ and hence $\gamma_\psi(C_n) = n - 1$. Further, obviously, $\gamma_\psi(C_n) \le n - 1$ for any acyclic graphoidal cover ψ of C_n. Hence $\gamma_{GC}(C_n) = n - 1$.

Now, for the acyclic graphoidal cover ψ = $E(C_n)$, we have $\gamma_\psi(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$ so that $\gamma_{gc}(C_n) \le \lceil \frac{n}{3} \rceil$. Now, let ψ be any acyclic graphoidal cover of C_n with $|\psi| = r > 2$. Then $C_n(\psi)$ is isomorphic to $C_r \cup (n - r)K_1$. Hence $\gamma_\psi(C_n) = \lceil \frac{r}{3} \rceil + (n - r) \ge \lceil \frac{n}{3} \rceil$. Hence $\gamma_{gc}(C_n) = \lceil \frac{n}{3} \rceil$.

Theorem 1.10. . Let G = W_{n+1} be the wheel on n+1 vertices. Then $\gamma_{gc}(G) = 1$ and $\gamma_{GC}(G) = \lceil \frac{n}{2} \rceil$.

Proof. Since $\gamma(G) = 1$, it follows that $\gamma_{gc}(G) = 1$.

Let V (G) = $\{v, v_1, v_2, \ldots, v_n\}$ and let E(G) = $\{vv_i : 1 \le i \le n\} \cup \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_1, v_n\}$. Now, let $P_1 = (v, v_{2i-1}, v_{2i})$, where $1 \le i \le \lfloor \frac{n}{2} \rfloor$ and let $Q = (v, v_n, v_1)$. Let $\psi = \{P_1, P_2, \ldots, P_{\lfloor \frac{n}{2} \rfloor}\} \cup S$ if n is even and let $\psi = \{P_1, P_2, \ldots, P_{\lfloor \frac{n}{2} \rfloor}, Q\} \cup S$ if n is odd, where S is the set of edges of G not covered by the paths P_i's. Now, if n is even, then G(ψ) is isomorphic to $S(K_{1, \frac{n}{2}})$ and if n is odd, then G(ψ) is isomorphic to the graph obtained from $K_{1, \lceil \frac{n}{2} \rceil}$ by subdividing all the edges of $K_{1, \lceil \frac{n}{2} \rceil}$ except one edge. Hence $\gamma_\psi(G) = \lceil \frac{n}{2} \rceil$, so that $\gamma_{GC}(G) \ge \lceil \frac{n}{2} \rceil$. Now, let ψ be any acyclic graphoidal cover of G. Since $\delta(G) \ge 3$, we have G(ψ) has no isolates and hence $\gamma_\psi(G) \le \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$. Hence $\gamma_{GC}(G) \le \lceil \frac{n}{2} \rceil$.

Thus $\gamma_{GC}(G) = \lceil \frac{n}{2} \rceil$.

The following theorem gives a domination chain for gc-domination.

Theorem 1.11. *For any graph G,* $ir_{gc}(G) \le \gamma_{gc}(G) \le i_{gc}(G) \le \beta_{0\,gc}(G) \le \Gamma_{gc}(G) \le IR_{gc}(G)$.

Proof. Let G be any graph. Let ψ be an acyclic graphoidal cover of G, such that $\gamma_{gc}(G) = \gamma_\psi(G)$. Now, $ir_\psi(G) \le \gamma_\psi(G) = \gamma_{gc}(G)$. Further $ir_{gc}(G) \le ir_\psi(G)$ and hence $ir_{gc}(G) \le \gamma_{gc}(G)$.

Now, choose an acyclic graphoidal cover ψ of G such that $i_{gc}(G) = i_\psi(G)$.Then $\gamma_{gc}(G) \le \gamma_\psi(G) \le i_\psi(G) = i_{gc}(G)$ and hence $\gamma_{gc}(G) \le i_{gc}(G)$. By a similar argument, we can prove that $i_{gc}(G) \le \beta_{0\,gc}(G)$, $\beta_{0\,gc}(G) \le \Gamma_{gc}(G)$ and $\Gamma_{gc}(G) \le IR_{gc}(G)$.

Hence

$ir_{gc}(G) \le \gamma_{gc}(G) \le i_{gc}(G) \le \beta_{0\,gc}(G) \le \Gamma_{gc}(G) \le IR_{gc}(G)$.

The following theorem gives a domination chain for GC-domination.

Theorem 1.12. *For any graph G,* $ir_{GC}(G) \le \gamma_{GC}(G) \le i_{GC}(G) \le \beta_{0\,GC}(G) \le \Gamma_{GC}(G) \le IR_{GC}(G)$.

Proof. Let G be any graph. Choose an acyclic graphoidal cover ψ of G such that $ir_{GC}(G) = ir_\psi(G)$.

Then $ir_{GC}(G) = ir_\psi(G) \le \gamma_\psi(G) \le \gamma_{GC}(G)$ and hence $ir_{GC}(G) \le \gamma_{GC}(G)$.

By a similar argument we can prove that $\gamma_{GC}(G) \le i_{GC}(G)$, $i_{GC}(G) \le \beta_{0\,GC}(G)$, $\beta_{0\,GC}(G) \le \Gamma_{GC}(G)$ and $\Gamma_{GC}(G) \le IR_{GC}(G)$.

Hence

$ir_{GC}(G) \le \gamma_{GC}(G) \le I_{GC}(G) \le \beta_{0\,GC}(G) \le \Gamma_{GC}(G) \le IR_{GC}(G)$.

Theorem 1.13. For any graph G,

$$\gamma_{gc}(G) \le 2\, ir_{gc}(G) - 1.$$

Proof. Let G be any graph G and let ψ be any acyclic graphoidal cover of G. We have, for any graph G $\gamma(G) \le 2\, ir(G) - 1$. It follows from this $\gamma_\psi(G) \le 2\, ir_\psi(G) - 1$. Now choose an acyclic graphoidal cover ψ of G such that $ir_\psi(G) = ir_{gc}(G)$. Then $\gamma_{gc}(G) \le \gamma_\psi(G) \le 2\, ir_\psi(G) - 1 = 2\, ir_{gc}(G) - 1$.

Thus $\gamma_{gc}(G) \le 2\, ir_{gc}(G) - 1$

Corollary 1.14. For any graph G,

$$\frac{1+\gamma_{gc}(G)}{2} \le ir_{gc}(G) \le \gamma_{gc}(G).$$

Theorem 1.15. For any graph G,

$$\frac{\gamma_{gc}(G)}{2} < ir_{gc}(G) \le \gamma_{gc}(G) \le 2\, ir_{gc} - 1.$$

Proof. Let G be any graph and ψ be any acyclic graphoidal cover of G.

It follows from Theorem 1.13 and Corollary 1.14 that $\frac{\gamma_{gc}(G)}{2} < ir_{gc}(G) \le \gamma_{gc}(G) \le 2\, ir_{gc} - 1$.

CONCLUSION

I found the parameters $\gamma_{gc}(G)$ and $\gamma_{GC}(G)$ for standard graphs and we can find the same parameters for any graph..

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