Some Restricted Plane partitions and Associated Lattice Paths

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Abstract. *Anand and Agarwal, (Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No.1, February 2012, 23-39) defined a lattice path representation for partitions with n copies of n using a class of weighted lattice paths called associated lattice paths. In this paper, using this correspondence between associated lattice paths and partitions with n copies of n and Agarwal's version of Bender and Knuth bijection (Bender and Knuth, J. Combin. Theory (A), 13, 1972, 40-54) between partitions with n copies of n and plane partitions, a three-way correspondence between a class of plane partitions, a class of partitions with n copies of n and a class of associated lattice paths is established.*

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I. INTRODUCTION

We first recall the definitions and results we shall use in this paper:

Definition1.1 (Agarwal and Andrews [5]). A partition with n copies of n is a partition in which a part of size n, $n \geq 0$ can come in n different colors denoted by subscripts: $n_1, n_2, \ldots. n_n$.

For example, Partitions of 2 with n copies of n are: 2_1, 2_2, $1_1 + 1_1$

Definition1.2 Plane Partition (Macmahon, [9]). A plane partition of a positive integer ν is an array of non-negative integers

$$\begin{matrix} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \ldots & \ldots & \ldots & \end{matrix}$$

for which $\sum_{i,j} n_{ij} = \nu$

and rows and columns are in non-increasing order.

If in a plane partition π of a positive integer v, there are λ_i parts in the i^{th} row of π so that, for some r, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_r > \lambda_{r+1} = 0$, then we call the partition $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_r$ of the integer $p = \lambda_1 + \lambda_2 + \lambda_3 \ldots + \lambda_r$, the shape of π.

Next, we recall the following description of associated lattice paths defined by Anand and Agarwal [4].

All paths are of finite length lying in the first quadrant. They will begin on the y- axis and terminate on the x-axis. Only three moves are allowed at each step.

northeast: from (i, j) to $(i + 1, j + 1)$,

southeast: from (i, j) to $(i + 1, j - 1)$, only allowed if $j > 0$,

horizontal: from (i, j) to $(i + 1, j)$, only allowed when the first step is preceded by a northeast step and the last is followed by a southeast step.

The following terminology will be used in describing associated lattice paths:

Truncated Isosceles Trapezoidal Section (TITS): A section of path which starts on x-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on x-axis forms what we call a Truncated Isosceles Trapezoidal Section.

Since the lower base lies on x-axis and is not a part of the path, hence the term truncated.

Slant Section (SS): A section of path consisting of only southeast steps which starts on the y-axis (origin not included) and ends on the x-axis.

Height of a slant section is 't' if it starts from $(0,t)$. Clearly, a path can have an SS only in the beginning.

A lattice path can have at most one SS.

Weight of a TITS: To define this, we shall represent every TITS by an ordered pair $\{a,b\}$ where a denotes its altitude and b the length of the upper base.

Weight of the TITS with ordered pair $\{a,b\}$ is a units.

Weight of a lattice path is the sum of weights of its TITSs.

Slant Section is assigned weight zero.

For example, in Figure 1, the associated lattice path has one SS of height 1 and one TITS with ordered pair {2, 3} and its weight is 2 units.

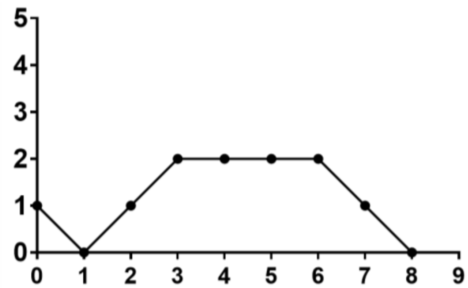


Fig: 1

Agarwal [3] established a bijection $\psi.\varphi$ between n-color partitions and plane partitions. For the clarity of our presentation, we shall first reproduce the bijection $\psi.\varphi$ here. In $\psi.\varphi$, φ is due to Bender and Knuth [8] and is the 1-1 correspondence of the following theorem:

Theorem 1.1 There is one to one correspondence between

1. the set of k x k matrices with non-negative integer entries
2. the set of all lexicographically ordered sequences of ordered pairs of integers, each $\leq k$
3. the set of ordered pairs (π_1, π_2) of column strict plane partitions of same shape in which each entry doesn't exceed k.

A different version of this theorem is also found in literature (cf. Stanley [11, 7.20]).

Theorem 1.2(Bender and Knuth). There is one to one correspondence between plane partitions of ν, on the one hand, and infinite matrices $a_{i,j}$ ($i,j \geq 1$) of non-negative integer entries which satisfy

$$\sum_{r\geq 1}\left\{\sum_{i+j=r+1} a_{ij}\right\} = \nu$$

on the other.

In the sequel, we shall call images $\varphi(\pi)$ BK_ν- matrices (B for Bender and K for Knuth). Although, these matrices are infinite matrices, but we represent them by largest possible square matrices containing at least one non-zero entry in the last row (or in the last column).

Thus, for example, we'll represent six BK_3- matrices by

$$3\ ,\ \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\ ,\ \begin{matrix}1 & 1\\ 0 & 0\end{matrix}\ ,\ \begin{matrix}0 & 0\\ 0 & 1\end{matrix}\ ,\ \begin{matrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix}\ ,\ \begin{matrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{matrix}$$

Definition (Agarwal). We define a matrix $E_{i,\,j}$ as an infinite matrix whose $(i,j)^{\text{th}}$ entry is 1 and the other entries are all zeros. We call $E_{i,\,j}$ distinct parts of a BK_ν-matrix.

Now we define the mapping ψ as follows:

Let $\Delta = a_{11}E_{1,1} + a_{12}E_{1,2} + \ldots + a_{21}E_{2,1} + a_{22}E_{2,2} + \ldots$

be a BK_ν- matrix where a_{ij} are non-negative integers which denote the multiplicities of $E_{i,\,j}$.

We map each part $E_{p,\,q}$ of Δ to a single part m_i of an n-color partition of ν. The map denoted by ψ is defined as

$$\psi\colon E_{p,\,q} \rightarrow (p + q - 1)_p, \qquad (1.1)$$

and the inverse mapping ψ^{-1} is easily seen to be

$$\psi^{-1}\colon m_i \rightarrow E_{i,\, m-i+1}. \qquad (1.2)$$

Under this mapping, we see that each BK_ν matrix uniquely corresponds to an n-color partition of ν and vice versa. The composite of two mappings φ and ψ denoted by $\psi.\varphi$ is clearly a bijection between plane partitions of ν on one hand, and n-color partitions of ν, on the other.

II. Associated Lattice Paths and Plane Partitions

In this section, we establish a one to one correspondence between a class of plane partitions and a class of partitions with n copies of n. Also, since partitions with n copies of n are in one to one correspondence with a class of associated lattice paths, we also obtain a lattice path representation for this new class of plane partitions. We shall prove the following result:

Theorem 2.1 Let $P(\nu)$ denote the number of partitions of ν with n copies of n of the form

$$\sum_i (a_i)_{b_i}$$

such that

(a.) $a_1 \geq a_2 \geq a_3 \geq \ldots\ldots\ldots \geq a_r$

(b.) $b_1 \geq b_2 \geq b_3 \geq \ldots\ldots\ldots \geq b_r$

(c.) $a_1 - b_1 \geq a_2 - b_2 \geq a_3 - b_3 \geq \ldots\ldots\ldots \geq a_r - b_r$

(d.) If $(a_i) = (a_j)$, then $(b_i) = (b_j)$

Let $Q(\nu)$ denote the number of plane partitions of ν of the form

$$\pi = \begin{array}{ccc} l_{11} & l_{12} & \cdots \\ l_{21} & l_{22} & \cdots \\ \vdots & \vdots & \end{array}$$

such that $l_{ij} \leq 1 \ \forall \ i \geq 2$.

Then, $P(\nu) = Q(\nu) \ \forall \ \nu$.

Example. $P(4) = 12$, since the relevant partitions are

4_1, 4_2, 4_3, 4_4, $3_1 + 1_1$, $3_2 + 1_1$, $3_3 + 1_1$, $2_1 + 2_1$, $2_2 + 2_2$, $2_1 + 1_1 + 1_1$, $2_2 + 1_1 + 1_1$, $1_1 + 1_1 + 1_1 + 1_1 + 1_1$

Also, $Q(4) = 12$, in this case the relevant partitions are

4,	3 1,	3,	2 2,	2 1 1,	2 1,	1 1,	2,	1,	1 1 1 1,	1 1 1,	1 1
		1			1	1	1	1		1	1 1
						1	1	1			
								1			

Proof of Theorem 2.1 Let $w = (a_1)_{b1} + (a_2)_{b2} + \cdots + (a_r)_{br}$ be an n-color partition enumerated by $P(\nu)$, that is,

$a_1 \geq a_2 \geq a_3 \geq \ldots\ldots\ldots \geq a_r$,

$b_1 \geq b_2 \geq b_3 \geq \ldots\ldots\ldots \geq b_r$ and

$a_1 - b_1 \geq a_2 - b_2 \geq a_3 - b_3 \geq \ldots\ldots\ldots \geq a_r - b_r$, $\forall \ 1 \leq i \leq r$.

Corresponding BK_ν- matrix is

$$\Delta = E_{b_1, a_1 - b_1 + 1} + E_{b_2, a_2 - b_2 + 1} + \cdots + E_{b_r, a_r - b_r + 1}$$

where $b_i \geq b_{i+1}$ and $a_i - b_i + 1 \geq a_{i+1} - b_{i+1} + 1 \ \forall \ 1 \leq i \leq r - 1$.

This corresponds to pair of sequences

$$\begin{array}{ccccccc} b_1 & b_2 & . & . & . & b_r \\ a_1 - b_1 + 1 & a_2 - b_2 + 1 & . & . & . & a_r - b_r + 1 \end{array}$$

This pair of sequences corresponds to pair of column strict plane partitions (π_1, π_2) of same shape. The lower sequence corresponds to π_1 and upper sequence corresponds to π_2. Since the lower sequence is non-increasing,

$$\pi_1 = a_1 - b_1 + 1 \quad a_2 - b_2 + 1 \ldots \quad a_r - b_r + 1$$

and π_2 is of same shape. So $\pi_2 = b_1 \, b_2 \ldots b_r$.

This pair (π_1, π_2) corresponds to plane partition

$$\pi = \begin{array}{ccc} l_{11} & l_{12} & \cdots \\ l_{21} & l_{22} & \cdots \\ \vdots & \vdots & \end{array}$$

where $l_{1j} = a_j - b_j + 1$ and in $j^{(th)}$ column $l_{ij} = 1 \ \forall \ 2 \leq i \leq b_j - 1$ where $1 \leq j \leq r$ This plane partition of ν is enumerated by $Q(\nu)$. Hence, the result.

Note. Let w be a partition enumerated by $P(\nu)$ and let its corresponding plane partition be π. Then number of parts in w equals the number of columns of π and number of entries in j^{th} row of π equals b_j.

For the next result, we define the following:

Definition 2.1 The mass of any TITS with ordered pair $\{a,b\}$ is defined to be $a - b$.

In [4], it was proved that partitions of ν with n copies of n are in one to one correspondence with associated lattice paths of weight ν such that for any TITS with ordered pair $\{a,b\}$, b doesn't exceed a, TITS arranged in order of non-decreasing weights and not having any SS. In view of this result and the result proved above, following result can be proved easily. Here, we also provide a direct bijection between associated lattice paths enumerated by $C(\nu)$ and plane partitions enumerated by $Q(\nu)$

Theorem 2.2 Let $C(\nu)$ denote the number of associated lattice paths of weight ν such that

(a.) For any TITS with ordered pair $\{a,b\}$, b doesn't exceed a

(b.) TITS are arranged in order of non-decreasing weights

(c.) there is no SS

(d.) if weight of a TITS is greater than the other TITS, then its mass is either greater than or equal to the mass of the other

(e.) ITS with equal weights will have equal mass.

Then $P(\nu) = Q(\nu) = C(\nu) \ \forall \ \nu$.

Direct Bijection: Consider an associated lattice path enumerated by $C(\nu)$. Each TITS of this path corresponds to a column of plane partition enumerated by $Q(\nu)$ arranged in decreasing order of weights of corresponding TITS of the path. A TITS with ordered pair $\{a_i,b_i\}$corresponds to column of plane partition with first entry $a_i - b_i + 1$ followed by 1 in rest of $b_i - 1$ entries of that column. Thus, we get a plane partition enumerated by $Q(\nu)$. Conversely, each column of plane partition corresponds to a TITS with ordered pair $\{a_i,b_i\}$

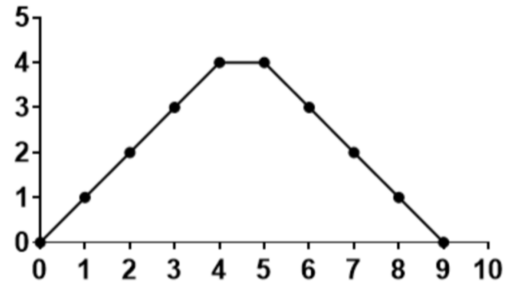
where b_i represents the number of entries in that column and a_i represents the first entry of that column minus number of entries in that column plus 1. This gives required path enumerated by $C(\nu)$.

Example: $P(4) = Q(4) = C(4) = 12$

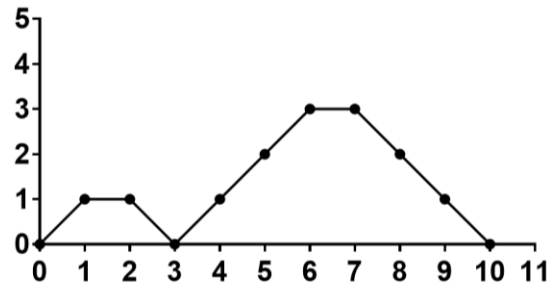
Following series of figures show one to one correspondence between plane partitions enumerated by $Q(4)$, partitions with n copies of n enumerated by $P(4)$ and lattice paths enumerated by $C(4)$

PLANE PART. EN. BY $Q(4)$ ⟷ COLORED PART. EN. BY $P(4)$ ⟷ ASSOCIATED LATTICE PATHS EN. BY $C(4)$

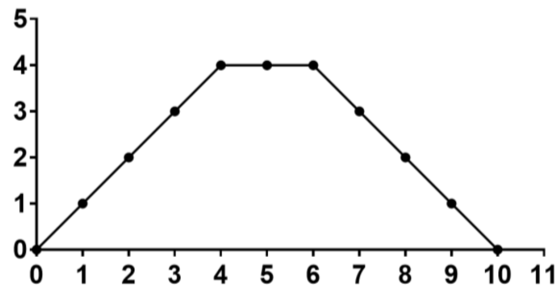
4 ⟷ 4_1 ⟷



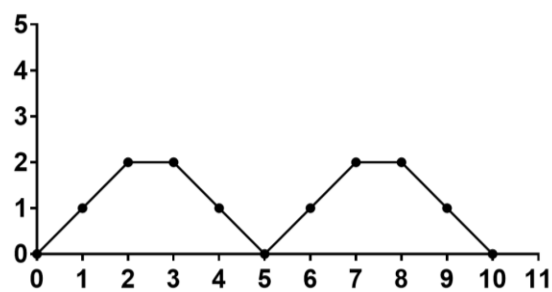
$3 \;\; 1$ ⟷ $3_1 + 1_1$ ⟷



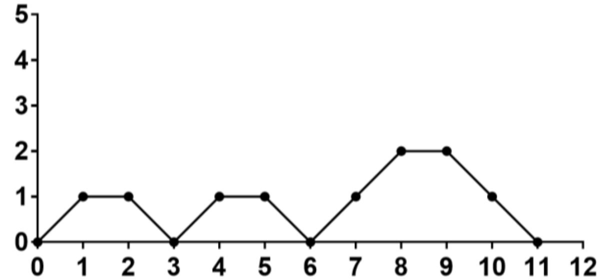
$\begin{matrix} 3 \\ 1 \end{matrix}$ ⟷ 4_2 ⟷



$2 \;\; 2$ ⟷ $2_1 + 2_1$ ⟷

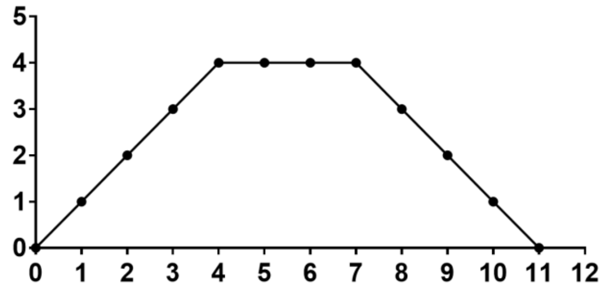


2 1 1 ⟷ $2_1 + 1_1 + 1_1$ ⟷

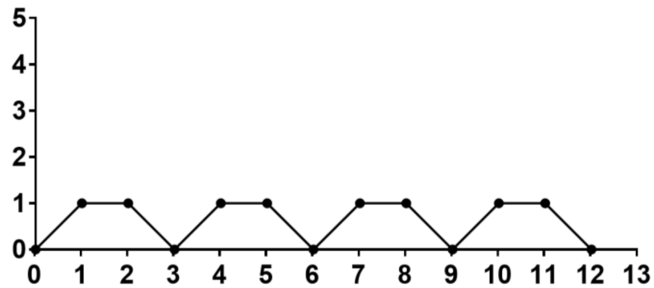


2 1
1 ⟷ $3_2 + 1_1$ ⟷

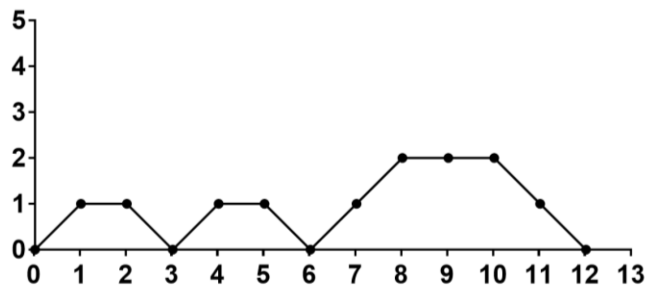
2
1
1 ⟷ 4_3 ⟷

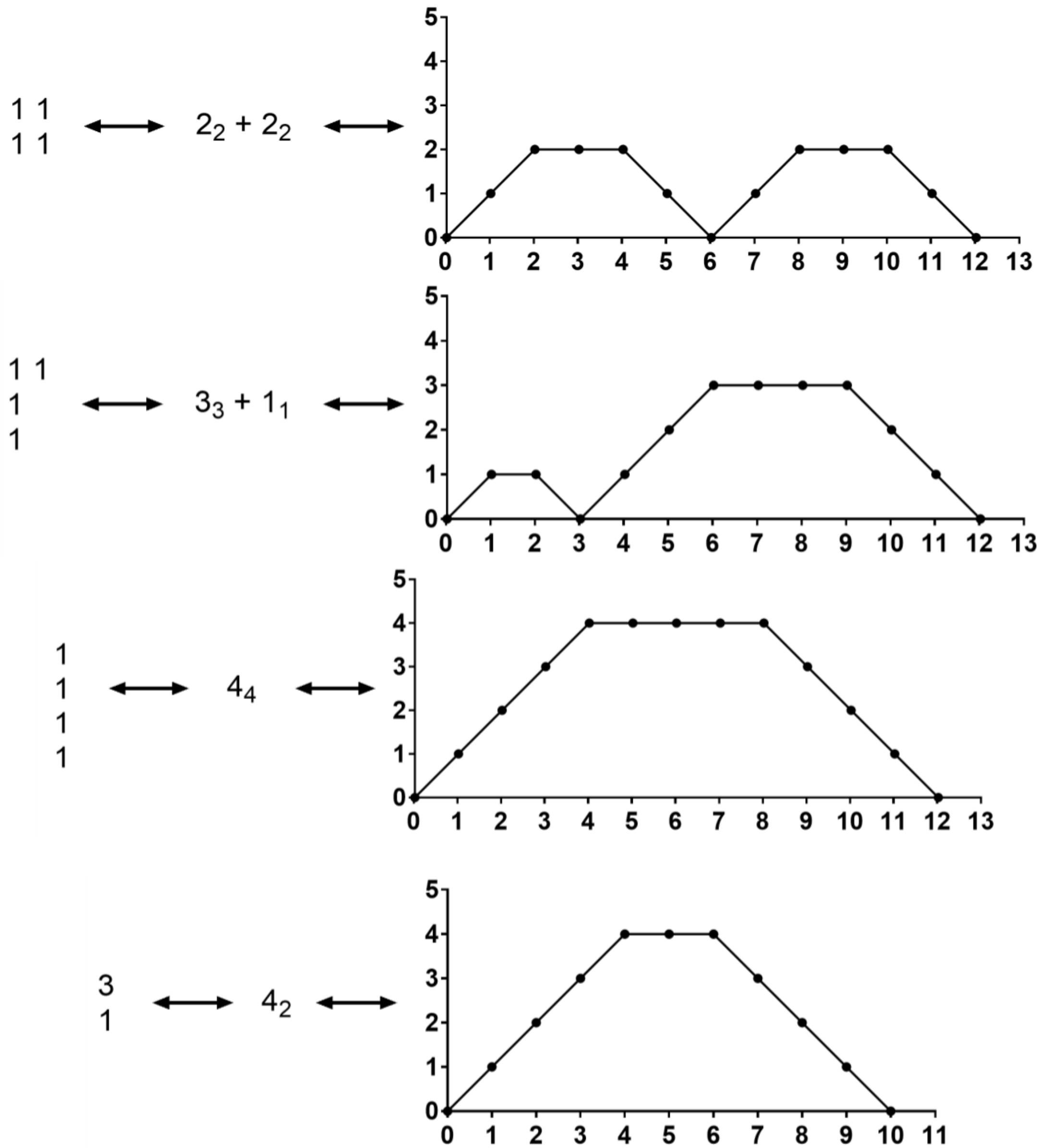


1 1 1 1 ⟷ $1_1 + 1_1 + 1_1 + 1_1$ ⟷



1 1 1
1 ⟷ $2_1 + 1_1 + 1_1$ ⟷





III. CONCLUSION.

Theorem 2.1 gives us a direct correspondence between a class of plane partitions of an integer v and a class of partitions with n copies of n of the integer v. Also, since partitions with n copies of n have a lattice path representation. So, we obtain a lattice path representation for a class of restricted plane partitions. But there is another class of plane partitions for which lattice path representation is still to be found.

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