Combinatorial Approach to Legendre Polynomials

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Abstract — *Not all differential equations have solutions in terms of known functions. Some of these have solution in series. This series gives rise to some interesting results. The well known such differential equations are Legendr's and Bessle's equations.*

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INTRODUCTION

Legendre's polynomial of degree n is a solution of the differential equation $(1-x^2)y_2-2xy_1+n(n+1)y=0$, known as Legendre's differential equation of order n. and denoted by $P_n(x)$. Some of these polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3/2)x^2-1/2$. In the following paper, it is intended to study some properties of polynomials from combinatorial point of view for which the following results are needed.

1. $$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}(k+r)!}{k!} = 0 \qquad 0\le r<n$$

1(a) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^r = 0 \qquad 0\le r<n$$

2. $$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}(k+n)!}{k!} = (-1)^n n!$$

2(a) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^n = (-1)^n n!$$

3. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = 0\text{, n odd}$$

3(a) $$\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} = (-1)^m \frac{(2m)!}{m!^2}$$

4. $$\int_0^{\pi/2} Sin^{2m+1}\theta\, Cos^{2n+1}\theta\, d\theta = \frac{m!\,n!}{2(m+n+1)!}$$

Proof:

Consider $x^r(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{r+k}$ **r≤n**

Differentiating r times we get

$$\sum_{k=0}^{r} \binom{n}{k}(1+x)^{n-k}\binom{r}{k}x^k r! = \sum_{k=0}^{n} \frac{\binom{n}{k}(k+r)!}{k!} x^k$$

For x=-1, r<n (1) follows and for x = -1,r = n, we get (2)

Again putting r=0,1,2....,n-1 in (1) we get

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(k+1) = \cdots\ldots = 0$$

from which 1(a) follows.

Now (2) can be written as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}(k+1)(k+2)\ldots\ldots(k+n) = (-1)^n n!$$

Or $\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^n = (-1)^n n!$ **which is 2(a), all other terms vanish because of 1(a)**

(3) follows from $(x^2-1)^n = (1+x)^n(x-1)^n$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k}$$

by comparing coefficients of x^n **where n is odd and noting that left side contains even powers of x only.**

Putting n=2m and comparing coefficients of x^{2m} **3(a) is obtained.**

The well known result (4) follows from reduction formula for $\int_0^{\pi/2} Sin^p\theta\, Cos\theta^q\, d\theta$

Rodrigue's formula states that if $P_n(x)$ **is the Legendre Polynomial of degree n , then**

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Let $y = (x^2-1)^n = (x+1)^n(x-1)^n = uv$

where $u = (x+1)^n$ **and** $v = (x-1)^n$

$u_r = n!(x+1)^{n-r}/(n-r)! = \binom{n}{r} r!\,(x+1)^{n-r}$ **,**

u_r **denoting the** r^{th} **derivative of u w.r.t. x**

From Leibnitz theorem on the n^{th} **derivative of the product of two functions , we have**

$$y_n = \sum_{k=0}^{n} \binom{n}{k} u_{n-k}\, v_k$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}(n-k)!\binom{n}{k}k!\,(x+1)^{n-k}(x-1)^k$$

$$= n! \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^{n-k}(x-1)^k$$

And consequently

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^{n-k}(x-1)^k$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^{k}(x-1)^{n-k}$$

using $\binom{n}{k} = \binom{n}{n-k}$

Properties of Legender Polynomials:

(i) $P_n(-x) = (-1)^n P_n(x)$

Proof: $P_n(-x) = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 (-x+1)^{n-k}(-x-1)^k$

$$= \frac{(-1)^n}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}(x+1)^k$$

$$= (-1)^n P_n(x)$$

(ii) $P_n(0) = 0$ **for odd n**

Proof: $P_n(0) = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^{n-k}$

$$= \frac{(-1)^n}{2^n}\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = 0$$

(from 3)

(iii) $P_{2n}(0) = \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{n!^2}$

Proof: $P_{2n}(0) = \frac{1}{2^{2n}} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2$

and the result follows from 3(a)

(iv) $P'_n(1) = \frac{n(n+1)}{2}$

where (′) denotes differentiation w.r.t. x

Proof: $P'_n(x)$

$$= \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 \{(n-k)(x+1)^{n-k-1}(x-1)^k + k(x+1)^{n-k}(x-1)^{k-1}\}$$

Which gives

$$P'_n(1) = \frac{1}{2^n}\binom{n}{0}^2 (n)(2)^{n-1} + \frac{1}{2^n}\binom{n}{1}^2 (2)^{n-1}$$

$$= \frac{n(n+1)}{2}$$

Analogue of 1(a) and 2(a) for Legender Polynomials:

(v) $\int_0^{\frac{\pi}{2}} Cos^{2r}\theta\, P_n(Cos2\theta)Sin2\theta d\theta = 0, 0 \le r < n$

(vi) $\int_0^{\frac{\pi}{2}} Cos^{2n}\theta\, P_n(Cos2\theta)Sin2\theta d\theta = \frac{n!^2}{(2n+1)!}$

Proof: Using $1 + Cos2\theta = 2Cos^2\theta$ **and** $1 - Cos2\theta = 2Sin^2\theta$ **we have**

$$P_n(Cos2\theta) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 Sin^{2n-2k}\theta\, Cos^{2k}\theta$$

Now $\int_0^{\frac{\pi}{2}} Cos^{2r}\theta\, P_n(Cos2\theta)Sin2\theta d\theta$

$$= 2(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \int_0^{\frac{\pi}{2}} Sin^{2n-2k+1}\theta\, Cos^{2k+r+1}\theta d\theta$$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \frac{(n-k)!(k-r)!}{(n+r+1)!}$$

(From 4)

$$= (-1)^n \frac{n!}{(n+r+1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(k+r)!}{(k)!}$$

$= 0 \quad 0 \le r < n$ **from (1)**

And $\int_0^{\frac{\pi}{2}} Cos^{2n}\theta\, P_n(Cos2\theta)Sin2\theta d\theta$

$$= (-1)^n \frac{n!}{(2n+1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(k+n)!}{(k)!}$$

$= \frac{n!^2}{(2n+1)!}$ **from (2)**

Since $Sin^{2m-2r}\theta\, Cos^{2r}\theta$ **can be expressed in the form** $(-1)^{m-r} Cos^{2m}\theta + a\, Cos^{2(m-1)}\theta + b\, Cos^{2(m-2)}\theta + \ldots\ldots$

We also have:

(vii) $\int_0^{\frac{\pi}{2}} Sin^{2m-2r}\theta Cos^{2r}\theta\, P_n(Cos2\theta)Sin2\theta d\theta$

$= 0 \quad 0 \le r \le m < n$

(viii) $\int_0^{\frac{\pi}{2}} Sin^{2n-2r}\theta Cos^{2r}\theta\, P_n(Cos2\theta)Sin2\theta d\theta$

$= (-1)^{n-r} \frac{n!^2}{(2n+1)!}$

Orthogonal Properties:

(ix) $\int_0^{\frac{\pi}{2}} P_m(Cos2\theta)P_n(Cos2\theta)Sin2\theta d\theta = 0 \quad m \ne n$

(x) $\int_0^{\frac{\pi}{2}} P_n^2(Cos2\theta)Sin2\theta d\theta = \frac{1}{(2n+1)}$

Proof: $\int_0^{\frac{\pi}{2}} P_m(Cos2\theta)P_n(Cos2\theta)Sin2\theta d\theta$

$$= (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k}^2 \int_0^{\frac{\pi}{2}} Sin^{2m-2k}\theta Cos^{2k}\theta\, P_n(Cos2\theta)Sin2\theta d\theta$$

$= 0$ **for m<n from (viii)**

For n<m , the result follows by interchanging m and n.

$\int_0^{\frac{\pi}{2}} P_n^2(Cos2\theta)Sin2\theta d\theta$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \int_0^{\frac{\pi}{2}} Sin^{2n-2k}\theta Cos^{2k}\theta\, P_n(Cos2\theta) Sin2\theta d\theta$$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \frac{(-1)^{n-k} n!^2}{(2n+1)!} \quad \text{from (viii)}$$

$$= \frac{n!^2}{(2n+1)!} \frac{(2n)!}{n!^2}$$

$$= \frac{1}{(2n+1)}$$

Expansion of $P_n(x)$ in powers of x-1 and x+1

Consider $y = (x+1)^k (x-1)^{n-k} = uv$

where $u = (x+1)^k$ and $v = (x-1)^{n-k}$

$$u_r = \binom{k}{r} r!\,(x+1)^{k-r},$$

$$v_r = \binom{n-k}{r} r!\,(x-1)^{n-k-r}$$

$$\text{So } y_r = \sum_{p=0}^{r} \binom{r}{p} u_{r-p}\, v_p$$

$$= \sum_{p=0}^{r} \binom{r}{p} \binom{k}{r-p} (r-p)! \binom{n-k}{p} p!\,(x+1)^{k-r+p} (x-1)^{n-k-p}$$

$$= r! \sum_{p=0}^{r} \binom{k}{r-p} \binom{n-k}{p} (x+1)^{k-r+p} (x-1)^{n-k-p}$$

Put x = 1 and denoting the corresponding value by $y_r(1)$ we have

$$y_r(1) = r! \binom{k}{r-n+k} 2^{n-r}$$

If $P^{(r)}_n(x)$ denotes the r^{th} derivative of $P_n(x)$ w.r.t. x , then

$$P^{(r)}_n(1) = \frac{r!}{2^r} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{k}{r-n+k}$$

$$= \frac{n!}{2^r (n-r)!} \sum_{k=0}^{n} \binom{n}{k} \binom{r}{n-k}$$

$$= \frac{n!}{2^r (n-r)!} \sum_{k=0}^{n} \binom{n}{n-k} \binom{r}{n-k}$$

$$= \frac{n!}{2^r (n-r)!} \sum_{m=0}^{n} \binom{n}{m} \binom{r}{m}$$

$$= \frac{n!}{2^r (n-r)!} \frac{(n+r)!}{n!\,r!} \quad \text{from (iii)}$$

$$= \frac{(n+r)!}{2^r r!\,(n-r)!}$$

The Taylor expansion of $P_n(x)$ is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-1)^k}{k!} P^{(k)}_n(1)$$

$$= \sum_{k=0}^{n} \frac{(n+k)!}{2^k (k!)^2 (n-k)!} (x-1)^k$$

Changing x to –x , we have $P_n(x)$

$$= (-1)^n \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{2^k (k!)^2 (n-k)!} (x+1)^k$$

Adding the two expressions , we also have

$P_n(x)$

$$= \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{2^{k+1} (k!)^2 (n-k)!} \{(1-x)^k + (-1)^n (x+1)^k\}$$

It is easy to see that the numbers $P^{(k)}_n(1)$ satisfy the following relations:

$$P^{(k)}_{n-1}(1) = \frac{n-k}{n+k} P^{(k)}_n(1) \text{ and}$$

$$P^{(k+1)}_n(1) = \frac{(n+k+1)(n-k)}{2(k+1)} P^{(k)}_n(1)$$

These relations help us to obtain the recurrence relations of Legendre polynomials.

For instance, the following relation is derived:

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x)$$

Proof: we have $P'_{n+1}(x)$

$$= \sum_{k=0}^{n+1} \frac{k(x-1)^{k-1}}{k!} P^{(k)}_{n+1}(1)$$

$$= \sum_{k=1}^{n+1} \frac{(x-1)^{k-1}}{(k-1)!} P^{(k)}_{n+1}(1)$$

$$= \sum_{k=0}^{n} \frac{(x-1)^k}{k!} P^{(k+1)}_{n+1}(1)$$

$$= \sum_{k=0}^{n} \frac{(x-1)^k}{k!} \frac{(n+k+2)(n+k+1)}{2(k+1)} P^{(k)}_n(1)$$

Similary, $P'_{n-1}(x)$

$$= \sum_{k=0}^{n} \frac{(x-1)^k}{k!} \frac{(n-k)(n-k-1)}{2(k+1)} P^{(k)}_n(1)$$

Thus $P'_{n+1}(x) + P'_{n-1}(x)$

$$= \sum_{k=0}^{n} \frac{(x-1)^k}{k!} \left\{\frac{(n+k+1)(n+k+2)-(n-k)(n-k-1)}{2(k+1)}\right\} P^{(k)}_n(1)$$

$$= (2n+1) \sum_{k=0}^{n} \frac{(x-1)^k}{k!} P^{(k)}_n(1) = (2n+1) P_n(x)$$

CONCLUSIONS

Although there are other methods of deriving the above results, but making the use of combinatories simplifies the process of derivation. As is said that everything must be made as simple as possible (but not too simple).

REFERENCES

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