Some Results on Positive Elements in the Tensor Product of C*-algebras

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Abstract *—In this paper, we discuss different properties of positive elements in the tensor product of two C*-algebras. Considering the cone of positive elements in a C*-algebra $\mathcal{A}$, we define a cone norm on the set of Hermitian elements of $\mathcal{A}$. Some results regarding positive forms in the tensor product are also derived here.*

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1 Introduction

In 1943, Gelfand and Naimark introduced the notion of a C*-algebra,which is a Banach algebra with an involution * satisfying $\| a^* \| = \| a \|$ and $\| a^*a \| = \| a \|^2$. The term C*-algebra was introduced by I.E.Segal in 1947 to describe norm-closed subalgebras of $\beta(\mathcal{H})$, space of bounded linear operatorson some Hilbert space $\mathcal{H}$ (C stands for closed). In 1969, A.Guichardet [7] discussed about C*-tensor norms and the tensor product of C*-algebras. Kaijserand Sinclair [10] studied about the projective tensor product of C*-algebras in1984. Blecher in his paper [1], investigated the geometrical properties of algebra norms on the tensor product of C*- algebras. Keith Conard in his paper[3] discussed about the tensor products of linear maps.

In this paper, we derive some results on positive elements in the tensorproduct of two C*-algebras. We also define a cone norm on the set of Hermitianelements of a C*-algebra. Some results regarding positive forms in the tensorproduct of two C*-algebras are also discussed here.

Definition 1.1. [11]:Let$\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras, $\mathcal{A}\otimes\mathcal{B}$ denote the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$. Then $\mathcal{A}\otimes\mathcal{B}$ is a C*-algebra under the natural definitions:

$$(a\otimes b)(a'\otimes b') = aa'\otimes bb'$$

and involution$(\sum_i a_i \otimes b_i)^* = \sum_i a_i^* \otimes b_i^*$where $a, a', a_i \epsilon \mathcal{A}$ and $b, b', b_i \epsilon \mathcal{B}$.

Definition 1.2. [1]:If α is a norm on $\mathcal{A}\otimes\mathcal{B}$ then α is called a crossnorm if $\| a\otimes b \|_\alpha = \| a \| \| b \|$for $a \,\epsilon\, \mathcal{A}$ and $b \,\epsilon\, \mathcal{B}$.

Definition 1.3. [11]: A norm on a *-algebra $\mathcal{A}$that satisfies $\| a^*a \| = \| a \|^2$ for all a in $\mathcal{A}$ is called a C*-norm.

If α is an algebra norm defined on an algebra$\mathcal{A}$, we call α a C*-*norm* on $\mathcal{A}$ if there is an involution on the α-completion of $\mathcal{A}$ making it into a C*-algebra. If $\mathcal{A}$ and$\mathcal{B}$ are C*-algebras, then there are several norms α that turn $\mathcal{A}\otimes_\alpha\mathcal{B}$ (completion of $\mathcal{A}\otimes\mathcal{B}$with respect to α) into a C*-algebra.

Definition 1.4. [13]: Let $\mathcal{A}$ be a C*-algebra. An element $a \in \mathcal{A}$ is called

i) self adjoint or hermitian if $a = a^*$

ii) normal if $a^*a = aa^*$

iii) a projection if $a = a^2 = a^*$

iv) unitary if $a^*a = e = aa^*$ (e being the unit element of $\mathcal{A}$)

Definition 1.5. [14]: Let $\mathcal{A}$ be a C*-algebra. An element $a \in \mathcal{A}$ is called positive if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}^+$. It is denoted by $a \geq 0$.

The set of all positive elements of $\mathcal{A}$is denoted by $\mathcal{A}^+$. For a unital C*-algebra $\mathcal{A}$ with $a, b \in \mathcal{A}$, we write$a \geq b$ when a and b are self adjoint and$(a - b) \geq 0$. Then $\geq$ is a partial order on the set of self adjoint elements of $\mathcal{A}$.

Definition 1.6.[4]:Let $\mathcal{A}$ be a C*-algebra. A linear form (or functional) f on$\mathcal{A}$ is said to be positive if $f(x^*x) \geq 0$ for each$x \in \mathcal{A}$. A state on $\mathcal{A}$ isa continuous positive linear form f on $\mathcal{A}$ such that $\| f \| = 1$.

Definition 1.7. [13]: A trace on $\mathcal{A}$is a positive linear form f with $f(e) = 1$satisfying $f(ab) = f(ba)$for all $a, b \in \mathcal{A}$. This condition is usually called the trace property. The trace is said to be faithful if $f(a^*a) = 0$ occurs only for $a = 0$.

If $\mathcal{A}$ is commutative, every positive linear form is a trace.

Definition 1.8. [2] A positive linear form f on $\mathcal{A}$is called pure if for any positive linear form g on$\mathcal{A}$ satisfying$g \leq f$, we have $g = \alpha f$for some$\alpha \geq 0$. We call a positive form f (which is also a homomorphism) pure (uptohomomorphism), if for any positive linear form (also homomorphism)g on

$\mathcal{A}$ satisfying $g \leq f$, we have $g = \alpha f$ for some $\alpha \geq 0$.

2 MAIN RESULTS

First, we discuss some properties of positive elements in the tensor product of two concrete C*-algebras. For a Hilbert space $\mathcal{H}$, a sub-algebra of $\beta(\mathcal{H})$ which is closed under norm and under adjoint operation is called a concrete C*-algebra. If $\mathcal{A}$ is a concrete C*-algebra, it can be represented by $\mathcal{A} \hookrightarrow \beta(\mathcal{H})$. From the GNS theorem [14], we have, every C*-algebra is isometrically * isomorphic to a concrete C*-algebra. Here, we take two concrete C*-algebras $\mathcal{A}$ and $\mathcal{B}$ and a C*- norm α (which is also a cross norm) on $\mathcal{A} \otimes \mathcal{B}$.

Lemma2.1[14]: Let $\mathcal{A}$ be a C*-algebra and $a \in \mathcal{A}$. Then the following are equivalent:

i) $a \geq 0$.
ii) $a = b^*b$ for some $b \in \mathcal{A}$.
iii) $a = b^2$ for some self adjoint element $b \in \mathcal{A}$.

Lemma 2.2([14], [15]): Let $\mathcal{A}$ be a C*-algebra and $a, b \in \mathcal{A}$.

i) If $a \geq 0$ and $-a \geq 0$, then $a = 0$.
ii) If $a, b \geq 0$ and $ab = ba$, then $ab \geq 0$ and $a + b \geq 0$.
iii) If $\mathcal{A}$ is a unital C*-algebra, $a = a^*$ and $\| a \| \leq 2$, then $a \geq 0$ if and only if $\| a - e \| \leq 1$
iv) If a is positive and $a^n = a^m$ for some integers $0 \leq m \leq n$, then a is a projection.
v) Every projection is a positive element.
vi) Let a be a positive element in $\mathcal{A}$. Then for a given arbitrary positive integer n, there exists a unique positive element $b \in \mathcal{A}$ such that $b^n = a$.
vii) Let $t \in \mathbb{R}^+$ and $a, b \in \mathcal{A}^+$. Then $ta + b \in \mathcal{A}^+$.
viii) If $0 \leq a \leq b$ then $\| a \| \leq \| b \|$.
ix) If $a, b \in \mathcal{A}^+$ then
$$\| a - b \| \leq \max(\| a \|, \| b \|)$$

Regarding positive elements in $\mathcal{A} \otimes_\alpha \mathcal{B}$ we derive the following result:

Theorem 2.3 For $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

i) If $a \geq 0, b \geq 0$ then $a \otimes b \geq 0$ and the converse holds if $a \otimes b \neq 0$.
ii) If a and b are projections, then $a \otimes b$ is also a projection.
iii) If a and b are two normal elements, then $a \otimes b$ is also a normal element.
iv) Let $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$ be positive elements such that $b \geq a$ and $d \geq c$.Then
$$\| a \otimes b - c \otimes d \| \leq (\| b \| + \| d \|)^2.$$
v) For unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, if $a \otimes b \in \mathcal{A} \otimes_\alpha \mathcal{B}$ is such that $\max(\| a \|, \| b \|) \leq 2$, then
$$\| e_1 \otimes e_2 - a \otimes b \| \leq \begin{cases} 1 + 2 \| b \|, \text{if } a \text{ is positive} \\ 1 + 2 \| a \|, \text{if } b \text{ is positive} \end{cases}$$

(where e_1 and e_2 are unit elements of $\mathcal{A}$ and $\mathcal{B}$ respectively)

Proof:

i) As $a \geq 0$ and $b \geq 0$ so, $a = c^*c, b = d^*d$

for some $c \in \mathcal{A}, d \in \mathcal{B}$. So,
$$a \otimes b = c^*c \otimes d^*d = (c \otimes d)^*(c \otimes d)$$
for $c \otimes d \in \mathcal{A} \otimes_\alpha \mathcal{B}$ which implies $a \otimes b \geq 0$.
Conversely, let $a \otimes b \geq 0$ and $a \geq 0$ but $b \leq 0 \Rightarrow -b \geq 0$.
Therefore, $a \otimes (-b) \geq 0 \Rightarrow -a \otimes b \geq 0 \Rightarrow a \otimes b \leq 0$.
So, $a \otimes b = 0$, a contradiction.
Thus $a \otimes b \geq 0 \Rightarrow a \geq 0, b \geq 0$.

ii) a and b are projections implies $a = a2=a*$

and $b = b^2 = b^*$. Now,
$$a \otimes b = a^* \otimes b^* = (a \otimes b)^*$$
$$a \otimes b = a^2 \otimes b^2 = (a \otimes b)^2,$$
showing that $a \otimes b$ is a projection.

iii) As a and b are normal elements, $a^*a = aa^*$ and $b^*b = bb^*$.

Now,
$$(a \otimes b)(a \otimes b)^* = aa^* \otimes bb^* = a^*a \otimes b^*b = (a \otimes b)^*(a \otimes b)$$
Thus, $(a \otimes b)$ is also normal.

iv) $\| a \otimes b - c \otimes d \| \leq \| a \otimes b \| + \| c \otimes d \|$
$$= \| a \| \| b \| + \| c \| \| d \|$$
$$\leq \| b \| \| b \| + \| d \| \| d \|$$
(by Lemma 2.2(viii))
$$= \| b \|^2 + \| d \|^2$$
$$\leq (\| b \| + \| d \|)^2$$

v) Let $a \in \mathcal{A}$ be positive and $b \in \mathcal{B}$ be arbitrary. Then , $\| e_1 \otimes e_2 - a \otimes b \|$
$$= \| (e_1 - a) \otimes b + e_1 \otimes (e_2 - b) \|$$
$$\leq \| e_1 - a \| \| b \| + \| e_1 \| \| e_2 - b \|$$
$$\leq 1. \| b \| + \| e_2 - b \| \text{[by Lemma 2.2 (iii)]}$$
$$\leq 1 + 2 \| b \|$$

Similarly if $b \in \mathcal{B}$ is positive, then
$$\| e_1 \otimes e_2 - a \otimes b \| \leq 1 + 2 \| a \|$$
□

Next, we consider the set of Hermitian elements H (which is avector space over $\mathbb{R}$) on a commutative realC*-algebra $\mathcal{A}$. With the help of positive elements we show that it is a cone Banach space over $\mathcal{A}$. As a genealization of normed spaces, cone normedspaces play a very important role in different branches of functionalanalysis. In 2009, .M.E.Gordji et al introduced the notion of conenormed spaces.

Definition 2.4 [5]For a vector space E, a subset P (of E) is called a cone whenever

i) P is a closed, non empty set and $P \neq \{0\}$,
ii) $ax + by \in P$ for all $x, y \in P$ and $a, b \geq 0$,
iii) $P \cap (-P) = \{0\}$.

With respect to$P$ a partial ordering$\leq$ can be defined on E by$x \leq y$if and only if $(y - x) \in P$. The cone $P$is normal if there is a number $K \geq 1$such that for all $x, y \in E, 0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$.

Definition 2.5 [5]Let X be a real vector space. If the mapping $\|.\|_P : X \to E$ satisfies:

a) $\| x \|_P \geq 0 \ \forall \ x \in X$and $\| x \|_P = 0$ iff $x = 0$,
b) $\| \alpha x \|_P = |\alpha| \| x \|_P, \forall \ x \in X$ and $\alpha \in \mathbb{R}$,
c) $\| x + y \|_P \leq \| x \|_P + \| y \|_P, \forall \ x, y \in X$,

then $\|.\|_P$ is called a cone norm on X and $(X, \|.\|_P)$ is called a cone normed space over E.

Here, we define a map: $\|.\|_P : H \to \mathcal{A}$ such that

$$\| x \|_P = \begin{cases} x, \text{if } x \text{ is positive} \\ 0, \text{if } x = 0 \\ -x, \text{if } x \text{ is not positive} \end{cases}$$

Theorem 2.6: With respect to $\| g \|_P$, H is a cone Banach space over $\mathcal{A}$.

Proof: Let $x \in$ Hbe arbitrary and α be a scalar. Then

i) $\| x \|_P \geq 0 \ \forall \ x \in$ Hand $\| x \|_P = 0$ iff $x = 0$ (by definition)

ii)

Case I: Let $x \geq 0$
For $\alpha \geq 0, \| \alpha x \|_P = \alpha x = |\alpha| \| x \|_P$
For $\alpha \leq 0, \| \alpha x \|_P = -\alpha x = |\alpha| \| x \|_P$

Case II: Let $x \leq 0$
For $\alpha \geq 0, \| \alpha x \|_P = -\alpha x = \alpha(-x) = |\alpha| \| x \|_P$
For $\alpha \leq 0, \| \alpha x \|_P = \alpha x = (-\alpha)(-x) = |\alpha| \| x \|_P$

iii) Let $x, y \in$ H be arbitrary.

Case I: Let $x \geq 0$
For $y \geq 0, \| x + y \|_P = x + y = \| x \|_P + \| y \|_P$
For $y \leq 0$ and $x \geq -y$ $i.e., x + y \geq 0$,
$\| x \|_P + \| y \|_P = x - y \geq x + y = \| x + y \|_P$
$(-y \geq 0 \Rightarrow -2y \geq 0 \Rightarrow (x - y) - (x + y) \geq 0 \Rightarrow x - y \geq x + y)$
For $-y \geq 0$ and $-y \geq x$ $i.e., -(x + y) \geq 0$,
$\| x \|_P + \| y \|_P = x - y \geq -(x + y) = \| x + y \|_P$

$(x \geq 0 \Rightarrow 2x \geq 0 \Rightarrow (x - y) + (x + y) \geq 0)$
Case II: If $-x \geq 0, -y \geq 0$,
$\| x \|_P + \| y \|_P = (-x) + (-y) = -(x + y) = \| x + y \|_P$

Thus $\|.\|_P$ is a cone norm on H.
To show that $(\text{H}, \|.\|_P)$ is complete:
Let$\{x_n\}$ be a Cauchy sequence in H.As H $\subseteq \mathcal{A}$ and $\mathcal{A}$is complete, so $\{x_n\}$ converges to some x as$n \to \infty$
i.e., $\lim_{n\to\infty}(x_n - x) = 0 \Rightarrow \lim_{n\to\infty} \| (x_n - x) \|_P = 0$ (by definition of the cone norm).
Thus H is complete with respect to $\|.\|_P$, i.e., it is a cone BanachSpace.
□

In [8], H.L.Guang and Z.Xian, proved the following fixed pointtheorem in cone Banach spaces.

Theorem 2.7: Let $(X, \|.\|_P)$ be a complete cone normed space,P be a normal cone with normal constant K. Suppose the mapping: $T: X \to X$satisfies the contractive condition : $\| Tx - Ty \|_P \leq k \| x - y \|_P, \forall \ x, y \in X$, where$k \in [0,1)$ is a constant. Then Thas a unique fixed point in$X$.

Example 2.8 For the cone Banach space $(\text{H}, \|.\|_P)$, we define$Tx = \frac{x}{\alpha} (\alpha \geq 1), \ x \in$ H. Then

$\| Tx - Ty \|_P = Tx - Ty$, if $Tx - Ty$ is positive
$= \frac{x}{\alpha} - \frac{y}{\alpha} = \frac{x-y}{\alpha} = \frac{1}{\alpha} \| x - y \|_P$ (as $x - y \geq 0$)
$\| Tx - Ty \|_P = 0$, if $Tx - Ty = 0$
$= \frac{1}{\alpha} \| x - y \|_P$ (as $x - y = 0$)
$\| Tx - Ty \|_P = -(Tx - Ty)$, if $Tx - Ty$ is not positive
$= -\left(\frac{x}{\alpha} - \frac{y}{\alpha}\right) = -\frac{1}{\alpha}(x - y)$
$= \frac{1}{\alpha} \| x - y \|_P$ (as $-(x - y) \geq 0$)

Thus k$\| Tx - Ty \|_P \leq \frac{1}{\alpha} \| x - y \|_P$. So by the above theorem T has a unique fixed point in H.

Next we discuss some properties of positive linear forms on C*-algebras. From [12] we have,

Lemma 2.9: Let $\mathcal{A}$ be a C*-algebra. Then

i) Every positive linear form f on $\mathcal{A}$is bounded and has norm *f(e)* (if $\mathcal{A}$ is unital with unit *e*).

ii) If f is a positive linear form on $\mathcal{A}$, then $f(a^*) = \overline{f(a)}, |f(a)|^2 \leq \| f \| f(a^*a) \forall a \in \mathcal{A}$ and $|f(b^*a)|^2 \leq f(b^*b)f(a^*a)$.

iii) If f is a bounded linear form on a unital C*-algebra $\mathcal{A}$ then f is positive iff $f(e) = 1$.

iv) If f_1 and f_2 are positive linear forms on a unital C*-algebra, then $\| f_1 + f_2 \| = \| f_1 \| + \| f_2 \|$.

Lemma 2.10:[12] Let $\mathcal{B}$ be a C*-subalgebra of a C*-algebra $\mathcal{A}$ and suppose that f_0 is a positive linear form on $\mathcal{B}$.Then there isa positive linear form f on $\mathcal{A}$ extending f_0 such that $\| f \| = \| f_0 \|$.

For two unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{F}(\mathcal{A} \otimes_\alpha \mathcal{B})$, $\mathcal{F}(\mathcal{A})$ and $\mathcal{F}(\mathcal{B})$ be the sets of all positive linear forms on $\mathcal{A} \otimes_\alpha \mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ respectively. Here we find a relation between $\mathcal{F}(\mathcal{A} \otimes_\alpha \mathcal{B})$ and $\mathcal{F}(\mathcal{A})$ and $\mathcal{F}(\mathcal{B})$.

Theorem 2.11: Corresponding to a positive linearform $f \in \mathcal{F}(\mathcal{A} \otimes_\alpha \mathcal{B})$, there exist two positive linear forms $\tilde{f_1}$ and $\tilde{f_2}$ on $\mathcal{A}$ and $\mathcal{B}$ respectively.

Proof: We consider the set

$$I = \{x \in \mathcal{A} \otimes_\alpha \mathcal{B} : f(x^*y) + f(y^*x) = 0 \ \forall\, y \in \mathcal{A} \otimes_\alpha \mathcal{B}\}$$

Then I is a subspace of $\mathcal{A} \otimes_\alpha \mathcal{B}$, as for $x_1, x_2 \in I, \alpha, \beta \in \mathbb{K}$ and $y \in \mathcal{A} \otimes_\alpha \mathcal{B}$ arbitrary,

$$\begin{aligned} f\big((\alpha x_1 + \beta x_2)^* y + y^*(\alpha x_1 + \beta x_2)\big) &= f(\bar{\alpha} x_1^* y) + f(\bar{\beta} x_2^* y) + f(y^*\alpha x_1) + f(y^*\beta x_2) \\ &= f\big(x_1^*(\bar{\alpha} y)\big) + f\big((\bar{\alpha} y)^* x_1\big) + f\left(x_2^*(\bar{\beta} y)\right) + f\big((\bar{\beta} y)^* x_2\big) \\ &= 0 \quad \text{(using the definition of } I) \end{aligned}$$

Again, for $x \in I$ and $y, z \in \mathcal{A} \otimes_\alpha \mathcal{B}$ arbitrary,

$$\begin{aligned} f\big((zx)^*y + y^*(zx)\big) &= f(x^*z^*y + y^*zx) \\ &= f(x^*(z^*y) + (z^*y)^*x) \\ &= 0, \end{aligned}$$

showingthat I is also an ideal.

Let $g_1 \in \mathcal{A}^*$ and $g_2 \in \mathcal{B}^*$ (the dual spaces) be two homomorphisms. We define $h_1 : I \to \mathcal{A}$ by

$$h_1\Big(\sum_i p_i \otimes q_i\Big) = \sum_i g_2(q_i)\, p_i$$

Clearly h_1 is linear.

For $x = \sum_i p_i \otimes q_i$, $y = \sum_j a_j \otimes b_j \in I$,

$$\begin{aligned} h_1(xy) &= h_1\left(\sum_{i,j} p_i a_j \otimes q_i b_j\right) \\ &= \sum_{i,j} g_2(q_i b_j) p_i a_j \\ &= \sum_{i,j} g_2(q_i) g_2(b_j) p_i a_j \\ &= \left(\sum_i g_2(q_i)\, p_i\right)\left(\sum_j g_2(b_j)\, a_j\right) \\ &= h_1\left(\sum_i p_i \otimes q_i\right) h_1\left(\sum_j a_j \otimes b_j\right) = h_1(x) h_1(y), \end{aligned}$$

showing that h_1 is also a homomorphism.

Similarly, we can define (using g_1) a homomorphism $h_2 : I \to \mathcal{B}$.

Let $I_1 = h_1(I), I_2 = h_2(I)$. Then, I_1 and I_2 are also ideals of $\mathcal{A}$ and $\mathcal{B}$ respectively.

We define $f_1 : I_1 \to \mathbb{K}$ by $f_1(a_1) = f(a_1 \otimes e_2), a_1 \in I_1$ (e_2 being the unitelement of $\mathcal{B}$). Then, f_1 is linear.

$$\begin{aligned} f_1({a_1}^* a_1) &= f({a_1}^* a_1 \otimes e_2) \\ &= f\big((a_1 \otimes e_2)^*(a_1 \otimes e_2)\big) \\ &\geq 0 \ \text{(since } f \text{ is a positive form)}, \end{aligned}$$

which shows that f_1 is a positive linear form on I_1.

Now, using Lemma 2.10, f_1 can be extended to a positive linearform $\tilde{f_1} : \mathcal{A} \to \mathbb{K}$, where $\tilde{f_1}|_{I_1} = f_1$, i.e. $\tilde{f_1} \in \mathcal{F}(\mathcal{A})$ with $\| \tilde{f_1} \| = \| f_1 \|$.

Similarly, defining $f_2 : I_2 \to \mathbb{K}$ by $f_2(b_1) = f(e_1 \otimes b_1)$, $b_1 \in I_2$ (e_1 being the unitelement of $\mathcal{A}$, we can find $\tilde{f_2} \in \mathcal{F}(\mathcal{B})$ with $\tilde{f_2}|_{I_2} = f_2$ and $\| \tilde{f_2} \| = \| f_2 \|$. □

Theorem 2.12: If I contains the unit element, the following propertiesof f are inherited by f_1 and f_2:

(i) If f is a trace so are f_1 and f_2.
(ii) If f is faithful so are f_1 and f_2.
(iii) If f is state, f_1 and f_2 are also states.
(iv) If f is pure so are f_1 and f_2.

(If f is such that $Re\, f(x) = 0 \forall x \in \mathcal{A} \otimes_\alpha \mathcal{B}$, then $f(x + x^*) = 0$ so, I containsthe unit element, i.e. $I = \mathcal{A} \otimes_\alpha \mathcal{B}$ and accordingly I_1 and I_2 also contain the unit elements.)

Proof : i) $f_1(e_1) = f(e_1 \otimes e_2) = 1$

$$\begin{aligned} f_1(a_1 a_2) &= f(a_1 a_2 \otimes e_2) \\ &= f\big((a_1 \otimes e_2)(a_2 \otimes e_2)\big) \\ &= f\big((a_2 \otimes e_2)(a_1 \otimes e_2)\big) = f(a_2 a_1 \otimes e_2) \\ &= f_1(a_2 a_1), \end{aligned}$$

showing that f_1 is a trace. Similarly f_2 is also a trace.

ii) $$\begin{aligned} f_1({a_1}^* a_1) = 0 &\Rightarrow f({a_1}^* a_1 \otimes e_2) = 0 \\ &= f\big((a_1 \otimes e_2)^*(a_1 \otimes e_2)\big) = 0 \\ &\Rightarrow a_1 \otimes e_2 = 0 \Rightarrow \| a_1 \otimes e_2 \| = 0 \\ &\Rightarrow \| a_1 \| \| e_2 \| = 0 \Rightarrow \| a_1 \| = 0 \Rightarrow a_1 = 0, \end{aligned}$$

which implies f_1 is faithful. Similarly, f_2 is also faithful.

(iii) Clearly if f is continuous, f_1 and f_2 are also continuous. Also $\| f \| = 1$.
Now, $\| f_1 \| = f_1(e_1) = f(e_1 \otimes e_2) = \| f \| = 1$.
Similarly, $\| f_2 \| = 1$, which shows that f_1 and f_2 are states.
iv) If g is a positive linear form dominated by f, i.e, $g \leq f$ then $g = \alpha f$ for some $\alpha \geq 0$. Let $g_1 \leq f_1$.
We define $h: \mathcal{A} \otimes_\alpha \mathcal{B} \to \mathbb{K}$ by

$$h\left(\sum_i p_i \otimes q_i\right) = \sum_i g_1(p_i) f(e_1 \otimes q_i)$$

Then h is a positive linear form on $\mathcal{A} \otimes_\alpha \mathcal{B}$. Also, $h\left((\sum_i p_i \otimes q_i)^*(\sum_i p_i \otimes q_i)\right) = h\left(\sum_{i,j} p_i{}^* p_j \otimes q_i{}^* q_j\right)$

$$= h\left(\sum_i p_i{}^* p_i \otimes q_i{}^* q_i\right) + \frac{1}{2} h\left(\sum_{i,j,i\neq j} (p_i{}^* p_j \otimes q_i{}^* q_j + p_j{}^* p_i \otimes q_j{}^* q_i)\right)$$

$$= \sum_i g_1(p_i{}^* p_i) f(e_1 \otimes q_i{}^* q_i) + \frac{1}{2} \sum_{i,j,i\neq j} \left(g_1(p_i{}^* p_j) f(e_1 \otimes q_i{}^* q_j) + g_1(p_j{}^* p_i) f(e_1 \otimes q_j{}^* q_i)\right)$$

$$\leq \sum_i f_1(p_i{}^* p_i) f(e_1 \otimes q_i{}^* q_i) + \frac{1}{2} \sum_{i,j,i\neq j} \bar{\alpha}_{i,j} f(e_1 \otimes q_i{}^* q_j) + \alpha_{i,j} f(e_1 \otimes q_j{}^* q_i),$$

taking $\alpha_{i,j} = g_1(p_j{}^* p_i)$

$$= \sum_i f_1(p_i{}^* p_i) f(e_1 \otimes q_i{}^* q_i) + \frac{1}{2} \sum_{i,j,i\neq j} f((\alpha_{i,j} e_1 \otimes q_i)^* (e_1 \otimes q_j) + (e_1 \otimes q_j)^* (\alpha_{i,j} e_1 \otimes q_i))$$

$= \sum_i f_1(p_i{}^* p_i) f(e_1 \otimes q_i{}^* q_i) + \frac{1}{2}.0$ (by definition of I)

$$= \sum_i f(p_i{}^* p_i \otimes e_2) f(e_1 \otimes q_i{}^* q_i) + A,$$

(where $A = \frac{1}{2} \sum_{i,j,i\neq j} (f_1(p_i{}^* p_j) f(e_1 \otimes q_i{}^* q_j) + f_1(p_j{}^* p_i) f(e_1 \otimes q_j{}^* q_i)$,
which is also 0 by definition of I and as in the above argument, taking $\alpha_{i,j} = f_1(p_j{}^* p_i)$.)

$$= \sum_i f((p_i{}^* \otimes q_i{}^*)(p_i \otimes q_i)) + \frac{1}{2} \sum_{i,j,i\neq j} \left(f\left((p_i{}^* \otimes q_i{}^*)(p_j \otimes q_j)\right) + f\left((p_j{}^* \otimes q_j{}^*)(p_i \otimes q_i)\right)\right)$$

$$= f\left(\left(\sum_i p_i \otimes q_i\right)^* \left(\sum_i p_i \otimes q_i\right)\right)$$

Showing that $h \leq f$. So, $h = \alpha f$ for some $\alpha \geq 0$. Then ,

$$h\left(\sum_i p_i \otimes q_i\right) = \alpha f\left(\sum_i p_i \otimes q_i\right) = \alpha \sum_i f_1(p_i) f(e_1 \otimes q_i)$$

$$\Rightarrow \sum_i g_1(p_i) f(e_1 \otimes q_i) = \alpha \sum_i f_1(p_i) f(e_1 \otimes q_i)$$

$\Rightarrow \sum_i (g_1(p_i) - \alpha f_1(p_i)) f(e_1 \otimes q_i)) = 0$ for any

$\sum_i p_i \otimes q_i \in \mathcal{A} \otimes_\alpha \mathcal{B}$.
So, in particular, for $a \otimes e_2 \in I_1 \otimes \mathcal{B} \subseteq \mathcal{A} \otimes_\alpha \mathcal{B}$;

$$(g_1(a) - \alpha f_1(a))\, f(e_1 \otimes e_2) = 0$$

$$\Rightarrow (g_1(a) - \alpha f_1(a)) \| f \| = 0$$

$$\Rightarrow g_1(a) = \alpha f_1(a),$$

$a \in \mathcal{A}$ being arbitrary, we have, $g_1 = \alpha f_1$. Thus, f_1 is a pure positive form.
Similarly, we can show that f_2 is also pure.
Now we proceed for the converse part of Theorem2.11.

Theorem 2.13 Corresponding to two positive linear forms $f_1 \in \mathcal{F}(\mathcal{A})$ and $f_2 \in \mathcal{F}(\mathcal{B})$ there exists a positive form $\tilde{f}$ on $\mathcal{A} \otimes_\alpha \mathcal{B}$.
Proof: For $f_1 \in \mathcal{F}(\mathcal{A})$, we construct the set

$$I_1 = \{a_1 \in \mathcal{A}: f_1(a_1^* a + a^* a_1) = 0 \; \forall a \in \mathcal{A}\}$$

Then, I_1 is a subspace of $\mathcal{A}$, as for $x_1, x_2 \in I_1, \alpha, \beta \in \mathbb{K}$ and $y \in \mathcal{A}$ arbitrary,

$$f_1((\alpha x_1 + \beta x_2)^* y + y^*(\alpha x_1 + \beta x_2)) = 0$$

(as in linear Theorem 2.11),
Again, for $x \in I_1$ and arbitrary $y, z \in \mathcal{A}$;
$f_1((zx)^* y + y^*(zx)) = 0$, showing that I_1 is an ideal of $\mathcal{A}$.
Similarly, for $f_2 \in \mathcal{F}(\mathcal{B})$ we take

$$I_2 = \{b_1 \in \mathcal{B}: f_2(b_1^* b + b^* b_1) = 0 \; \forall b \in \mathcal{B}\},$$

which will be an ideal of $\mathcal{B}$. Let

$$I = \{\sum_i x_i \otimes y_i \in I_1 \otimes I_2 : \sum_{i,j,i\neq j} f_1(x_i{}^*) f_2(y_j) + \sum_{i,j,i\neq j} f_1(x_i) f_2(y_j{}^*) = 0\}$$

Clearly, for $p, q \in I$, and $\alpha, \beta \in \mathbb{K}, \alpha p + \beta q \in I$.
Also, for $p \in I,\; x \in \mathcal{A} \otimes_\alpha \mathcal{B}$,

$$px = \left(\sum_i p_i \otimes q_i\right)\left(\sum_j x_j \otimes y_j\right) = \sum_{i,j} p_i x_j \otimes q_i y_j$$

Now, each $p_i x_j \in I_1, q_i y_j \in I_2$. So

$$f_1\left((p_i x_j)^* a + a^*(p_i x_j)\right) = 0 \ \forall\, a \in \mathcal{A} \text{ and } \forall\, i,j$$

$$f_2\left((q_i y_j)^* b + b^*(q_i y_j)\right) = 0 \ \forall\, b \in \mathcal{B} \text{ and } \forall\, i,j$$

In particular, for $a = f_2(q_i y_j)e_1$, we get

$$f_1\left((p_i x_j)^* f_2(q_i y_j) + (f_2(q_i y_j))^*(p_i x_j)\right) = 0$$

$$\Rightarrow f_1\left((p_i x_j)^*\right) f_2(q_i y_j) + f_1(p_i x_j) f_2((q_i y_j)^*) = 0$$

This will hold for any pair (i,j). Hence $px \in I$, showing that I is an ideal.

We define $f: I \to \mathbb{K}$ by $f(\sum_i a_i \otimes b_i) = \sum_i f_1(a_i) f_2(b_i)$

Clearly, f is linear and

$$f\left(\left(\sum_i a_i \otimes b_i\right)^* \left(\sum_i a_i \otimes b_i\right)\right) = f\left(\sum_{i,j} a_i^* a_j \otimes b_i^* b_j\right) = \sum_{i,j} f_1(a_i^* a_j) f_2(b_i^* b_j)$$

$$= \sum_i f_1(a_i^* a_i) f_2(b_i^* b_i) + \frac{1}{2}\left(\sum_{i,j,i\neq j} f_1\left((a_j^* a_i)^*\right) f_2(b_i^* b_j) + f_1(a_j^* a_i) f_2\left((b_i^* b_j)^*\right)\right)$$

$$= \sum_i f_1(a_i^* a_i) f_2(b_i^* b_i) + 0 \geq 0,$$

which implies that f is a positive linear form on I.

Now, using Lemma 2.10 we get a positive linearform $\tilde{f}$ on $\mathcal{A} \otimes_\alpha \mathcal{B}$ so that $\tilde{f}|_I = f$ and $\|\tilde{f}\| = \|f\|$.

□

Theorem 2.14 If I_1 and I_2 contain the unit elements, (i.e, $I_1 = \mathcal{A}$ and $I_2 = \mathcal{B}$) the positive linear form f defined in the above theorem satisfies the following properties:

i) If f_1 and f_2 are traces then f is also a trace.

ii) If f_1 and f_2 are faithful so is f.

iii) If f_1 and f_2 are states then f is also a state.

iv) If f_1 and f_2 are pure then f is also pure.

Proof:

i) $f(e_1 \otimes e_2) = f_1(e_1) f_2(e_2) = 1$

$$f\left(\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes d_j\right)\right) = \sum_{i,j} f_1(a_i c_j) f_2(b_i d_j)$$

$= \sum_{i,j} f_1(c_j a_i) f_2(d_j b_i) = f((\sum_j c_j \otimes d_j)(\sum_i a_i \otimes$ *bi*)),

Which shows that f is a trace.

ii) $f((\sum_i a_i \otimes b_i)^*(\sum_i a_i \otimes b_i)) = 0$

$$\Rightarrow \sum_{i,j} f_1(a_i^* a_j) f_2(b_i^* b_j) = 0$$

$\Rightarrow \sum_i f_1(a_i^* a_i) f_2(b_i^* b_i) = 0$ (by the definition of f in theorem 2.13)

$$\Rightarrow f_1(a_i^* a_i) f_2(b_i^* b_i) = 0 \ \forall\, i$$

$$\Rightarrow f_1(a_i^* a_i) = 0 \text{ or } f_2(b_i^* b_i) = 0 \ \forall\, i$$

$$\Rightarrow a_i = 0 \text{ or } b_i = 0 \ \forall\, i$$

$$\Rightarrow \sum_i a_i \otimes b_i = 0,$$

showing that f *is* faithful.

iii) f is continuous if f_1 and f_2 are continuous. Also,

$$\| f \| = f(e_1 \otimes e_2) = f_1(e_1) f_2(e_2) = \| f_1 \| \| f_2 \| = 1,$$

which implies f is also a state.

iv) Let $g \leq f$, we define $h_1: \mathcal{A} \to \mathbb{K}$ by

$$h_1(a) = g(a \otimes e_2)$$

$$h_1(a^* a) = g(a^* a \otimes e_2) = g\left((a \otimes e_2)^*(a \otimes e_2)\right) \leq f\left((a \otimes e_2)^*(a \otimes e_2)\right) = f(a^* a \otimes e_2) = f_1(a^* a) f_2(e_2) = f_1(a^* a)$$

$\Rightarrow h_1 \leq f_1$.

So, $h_1 = \alpha f_1$ for some $\alpha \geq 0$.

Therefore, $h_1(a) = \alpha f_1(a)$

$$\Rightarrow g(a \otimes e_2) = \alpha f_1(a) f_2(e_2) = \alpha f(a \otimes e_2) \ \forall\, a \in \mathcal{A}.$$

Now, let $h_2: \mathcal{B} \to \mathbb{K}$ be definedby $h_2(b) = g(e_1 \otimes b)$

As above we can show that $h_2 \leq f_2$. So,

$$h_2 = \beta f_2 \text{ for some } \beta \geq 0.$$

Therefore, $h_2(b) = \beta f_2(b)$.

Now, for $a \otimes b \in I$,

$$g(a \otimes b) = g((a \otimes e_2)(e_1 \otimes b)) = h_1(a) h_2(b) = \alpha f_1(a) \beta f_2(b) = \alpha\beta f(a \otimes b),$$

showing that f is pure (upto homomorphism). □

Concluding Remark: We have derived different results regarding positive elements and positive forms in the tensor product of C*-algebras. In 2014, S.H.Jah and M.S.Ahmed [9] derived some results on positive-normal operators in semi-Hilbertian spaces. Considering this aspect, the following problem can be raised:

Using a positive-normal element and a positive form on each of the two C*-algebras$\mathcal{A}$ and $\mathcal{B}$, can we obtain a class of positive forms on their tensor product?

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