Modeling and Analysis of an SEIRS Epidemic Model with Non-monotonic Incidence Rate

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Abstract *- In this paper we investigate a SEIRS epidemic model with nonlinear saturated incidence rate* $\frac{kSI}{1+\alpha I^2}$. *According to different recovery rates, we use differential stability theory and the global stability of the disease-free equilibrium, and the existence and global stability of the endemic equilibrium proved by constructing a Lyapunov function. Some numerical simulations are given to illustrate the analytical results.*

Keywords: *epidemic model, equilibrium, Lyapunov function, reproduction number.*

2010 Subject Classification: 92D30, 93A30, 93D30, 34D23.

I. INTRODUCTION

Infectious diseases have tremendous influence on human life and they cause panic to mankind out of control. Mathematical models describing the population dynamics of infectious diseases play an important role towards a better understanding of epidemiological patterns and disease control for a long time. One of the main issues in the study of behavior of epidemics is the analysis of steady states of the model and their stability. In this process, the rate of incidence plays an crucial role.

The bilinear incidence rate βSI and the standard incidence rate is $\frac{\beta SI}{N}$, where N is the total population size and β is called daily contact rate have been used frequently used in classical epidemic models. In 1978, the general incidence rate $\frac{\lambda I^p S}{1+\alpha I^q}$ was proposed by Liu in 1986-87, Esteva and Matias [3] introduced the saturated incidence rate $\frac{\beta SI}{1+\alpha I}$ which tends to a saturation level when I gets large, βI measures the infecting force when the disease is entering a fully susceptible population, and $\frac{1}{1+\alpha I}$ measures the inhibition effect of the behavioral change of susceptible individuals when their number increases or from the crowding effect of the infective individuals. This incidence rate is more reasonable than the bilinear incidence rate because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters. It was used in many epidemic models afterwards. Xiao and Ruan, 2007 proposed an epidemic model with non-monotonic incidence rate $\frac{\lambda IS}{1+\alpha I^2}$. In epidemiology using a compartmental approach, one may assume that a susceptible individual first goes through a latent period (said to become exposed or in the class E) after infection, before becoming infectious. The resulting models are of SEIRS or SEIRSS types, respectively, depending on whether the acquired immunity is permanent or otherwise.

These types of models have attracted the attention of many authors and a number of papers have been published in this area. For example, Greenhalgh [4] considered an SEIRS model that incorporates density dependence in the death rate. Li and Muldowney [9] and Li et al. [10] studied the global dynamics of the SEIRS models with a non-linear incidence rate as well as standardized incidence rate. Li et al. [8] analyzed the global dynamics of the SEIRS model with vertical transmission and a bilinear incidence. Rinalid [11] analyzed epidemic models with latent period. In 2003, Zhang and Ma [16] analyzed the global dynamics of the SEIR model with saturating contact rate. All the models discussed above are of SEIR-type epidemic models, which are described by a system of ordinary differential equations.

Recently, many authors generalized new incidence rate like Kar and Batabyal [5] proposed an SIR model with non-monotonic incidence rate suggested by Xiao and Ruan [15] includes a treatment function and G. Ujjainker [14] generalized the model of Kar and Batabyal [5] with two parameters.

Motivated by the work of Xiao and Ruan [15] and Kar and Batabyal [5], in this paper, we are concerned an SEIRS epidemic model with the effect of the nonmonotonic incidence rate function. The purpose of this paper is to show that stability of an SEIRS epidemic model.

II. MATHEMATICAL MODEL FORMULATION

In this section, we describe the SEIRS epidemic model with saturated incidence rate and introduce some correlative definitions about differential and algebraic systems. The SEIRS model in epidemiology for the spread of an infectious disease is described by the following system of differential equations:

$$\left.\begin{aligned}\frac{dS}{dt} &= B - dS - \frac{kSI}{1+\alpha I^2} + vR\\ \frac{dE}{dt} &= \frac{kSI}{1+\alpha I^2} - (\varepsilon + d)E\\ \frac{dI}{dt} &= \varepsilon E - (\gamma + d)I\\ \frac{dR}{dt} &= \gamma I - (v+d)R\end{aligned}\right\} \qquad (1)$$

where, $S>0$, $E>0$, $I>0$ and $R>0$ denotes the fractions of the population that are susceptible, exposed, infectious, and recovered, respectively, with temporary immunity, becoming susceptible again where immunity is lost. Some notable features of the model: B is the recruitment rate of the population, ν is the rate of losing immunity at time t, ε is the rate of developing infectivity, γ is the recovery rate, d is the natural death rate of the population, k is the proportionality constant and α is the parameter measures of the psychological or inhibitory effect.

We considered a non monotonic saturated rate of the form $\frac{kSI}{1+\alpha I^2}$. This represents the inhibition effect of the behavioral change of the susceptible individuals where there is an increase in the number of infective individuals, we assume that the birth rate and death rate are not equal.

III. EQUILIBRIUM POINTS

When we put the time derivatives equal to zero, we get diseases free equilibrium (DFE) $P_0(S_0,E_0,I_0,R_0)=\left(\frac{B}{d},0,0,0\right)$. For the endemic equilibrium $P^*(S^*,E^*,I^*,R^*)$ we have following relations are mentioned below:

$$S^*=\frac{(\varepsilon+d)(\gamma+d)(1+\alpha I^2)}{\varepsilon k}, \quad E^*=\frac{(\gamma+d)I}{\varepsilon},$$

$R^*=\frac{\gamma I}{(v+d)}$ and I is given as a root of the quadratic equation

$$aI^2+bI+c=0$$

where

$$a=\alpha d(\varepsilon+d)(\gamma+d),$$
$$b=k(\varepsilon+d)(\gamma+d)-\frac{\varepsilon\gamma vk}{(v+d)},$$
$$c=d(\varepsilon+d)(\gamma+d)-\varepsilon Bk$$

Clearly, the above equation will have a positive root if $\Delta>0$ and $R_0>1$, where R_0 is the basic reproduction number given as follows:

$$R_0=\frac{\varepsilon Bk}{d(\varepsilon+d)(\gamma+d)}$$

Now, $$I^*=\frac{-\left[k\left\{(\varepsilon+d)(\gamma+d)-\frac{\varepsilon\gamma v}{(v+d)}\right\}\right]\pm\sqrt{\Delta}}{2\alpha d(\varepsilon+d)(\gamma+d)}$$

where,

$$\Delta=[k\{(\varepsilon+d)(\gamma+d)-\frac{\varepsilon\gamma v}{(v+d)}\}]^2+4[\alpha d^2(\varepsilon+d)^2\times(\gamma+d)^2(R_0-1)].$$

Lemma 3.1 The system (1) has a disease free equilibrium points if $N=\frac{B}{d}$.

Proof. Consider $N(t)=S(t)+I(t)+R(t)$

Then have $\frac{dN}{dt}=B-dN(t)$

Simple mathematical calculation shows that $\lim_{t\to\infty}N(t)=\frac{B}{d}$.

This implies the conclusion.

IV. STABILITY

4.1 Local Stability of Diseases Free Equilibrium

Let $x=S-S_0$, $E=E$, $I=I$ and $R=R$.

System (1) becomes,

$$\left.\begin{aligned}
\frac{dx}{dt}&=B-d\left(x+\frac{B}{d}\right)-kI\left(x+\frac{B}{d}\right)(1+\alpha I^2)^{-1}+vR\\
\frac{dE}{dt}&=kI\left(x+\frac{B}{d}\right)(1+\alpha I^2)^{-1}-(\varepsilon+d)E\\
\frac{dI}{dt}&=\varepsilon E-(\gamma+d)I\\
\frac{dR}{dt}&=\gamma I-(v+d)R
\end{aligned}\right\}\quad(2)$$

By linearizing (1) we have

$$\left.\begin{aligned}
\frac{dS}{dt}&=-\frac{kB}{d}I-dS+vR+\text{non linear terms}\\
\frac{dE}{dt}&=\frac{kB}{d}I-(\varepsilon+d)E+\text{non linear terms}\\
\frac{dI}{dt}&=\varepsilon E-(\gamma+d)I\\
\frac{dR}{dt}&=\gamma I-(v+d)R
\end{aligned}\right\}\quad(3)$$

This can be written in matrix from

$$\begin{pmatrix}\frac{dx}{dt}\\ \frac{dE}{dt}\\ \frac{dI}{dt}\\ \frac{dR}{dt}\end{pmatrix}=\begin{pmatrix}-d & 0 & \frac{-kB}{d} & v\\ 0 & -(\varepsilon+d) & \frac{kB}{d} & 0\\ 0 & \varepsilon & -(\gamma+d) & 0\\ 0 & 0 & \gamma & -(v+d)\end{pmatrix}\begin{pmatrix}x\\ E\\ I\\ R\end{pmatrix}$$

Then characteristic equation of the given matrix is

$$|A-\lambda I|=0$$

$$|A-\lambda I|=-(d+\lambda)(d+v+\lambda)[-(\varepsilon+d+\lambda)(\gamma+d+\lambda)+\frac{\varepsilon Bk}{d}]=0$$

Lemma 4.1 If $R_0<1$, then the disease free equilibrium P_0 is asymptotically stable, P_0 is stable, if $R_0=1$ and P_0 is unstable if $R_0>1$.

Proof. We shall check the stability of the diseases equilibrium P_0 from the model, and then the linearization of disease free equilibrium P_0 gives the following characteristics equation:

$$-(d+\lambda)(v+d+\lambda)\left[-(\varepsilon+d+\lambda)(\gamma+d+\lambda)+\frac{\varepsilon Bk}{d}\right]=0$$

from the above equation, it can be seen that

$\lambda_1=-d,\lambda_2=-(d+v)$ are two eigen values and they are always negative. To obtain other eigen values

$$-(\varepsilon+d+\lambda)(\gamma+d+\lambda)+\frac{\varepsilon Bk}{d}=0$$

$$(\varepsilon+d+\lambda)(\gamma+d+\lambda)-\frac{\varepsilon Bk}{d}=0$$

$$\lambda^2+(2d+\varepsilon+\gamma)\lambda+\varepsilon\gamma+\varepsilon d+\gamma d+d^2-\frac{\varepsilon Bk}{d}=0.$$

For negative roots we must have

$$d^2+\varepsilon\gamma+\varepsilon d+\gamma d-\frac{Bk\varepsilon}{d}>0.$$

That is, if $R_0<1$ then diseases free equilibrium P_0, is locally asymptotically stable, if $R_0=1$, one eigen values is zero and it is simple then P_0 is stable. If $R_0>1$, then equation has one positive root, then P_0 is unstable.

4.2 Global stability of Diseases Free equilibrium

Define Lyapunov function:

$$L=\varepsilon E+(\varepsilon+d)I$$

By differentiating equation, we have

$$L'=\varepsilon E'+(\varepsilon+d)I'$$

$$L'=\varepsilon\left[\frac{kSI}{1+\alpha^2}-(\varepsilon+d)E\right]+(\varepsilon+d)\left[\varepsilon E-(\gamma+d)I\right]$$

$$L'=\frac{\varepsilon kSI}{1+\alpha I^2}-(\varepsilon+d)(\gamma+d)I$$

$$L'=(\varepsilon+d)(\gamma+d)\left[\frac{\varepsilon kS}{(1+\alpha I^2)(\varepsilon+d)(\gamma+d)}-1\right]I$$

$$L'=(\varepsilon+d)(\varepsilon+\gamma)\left[\frac{R_0}{1+\alpha I^2}-1\right]I$$

If $I=0\Rightarrow L'=0$ but if $I\neq 0$ and $R_0<1$ then $L'<0$. Therefore the disease free equilibrium is globally asymptotically stable.

4.3 Local Stability of Endemic Equilibrium

Let $x=S-S^*, y=E-E^*, z=I-I^*, q=R-R^*$

$$\left.\begin{aligned}
\frac{dx}{dt}&=B-d(x+S^*)-\frac{k(x+S^*)(z+I^*)}{1+\alpha(z+I^*)^2}+v(q+R^*)\\
\frac{dE}{dt}&=\frac{k(x+S^*)(z+I^*)}{1+\alpha(z+I^*)^2}-(\varepsilon+d)(y+E^*)\\
\frac{dI}{dt}&=\varepsilon(y+E^*)-(\gamma+d)(z+I^*)\\
\frac{dR}{dt}&=\gamma(z+I^*)-(v+d)(q+R^*)
\end{aligned}\right\} \quad (5)$$

The resulting Jacobin matrix is

$$A=\begin{pmatrix} -(d+\frac{kI^*}{1+\alpha I^{*2}}) & 0 & -\frac{kS^*}{1+\alpha I^{*2}} & v \\ \frac{kI^*}{1+\alpha I^{*2}} & -(\varepsilon+d) & \frac{kS^*}{1+\alpha I^{*2}} & 0 \\ 0 & \varepsilon & -(\gamma+d) & 0 \\ 0 & 0 & \gamma & -(v+d) \end{pmatrix} \quad (6)$$

Its characteristics equation is

$$\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4=0$$

where, $a_1=(\varepsilon+\gamma+3d+v)+(d+\frac{kI^*}{1+\alpha I^{*2}})$

$$a_2=(\varepsilon+d)(\gamma+d)+(\varepsilon+\gamma+2d)(d+v)-\frac{\varepsilon kS^*}{1+\alpha I^{*2}}+(d+\frac{kI^*}{1+\alpha I^{*2}})(\varepsilon+\gamma+3d+v)$$

$$a_3=(\varepsilon+d)(\gamma+d)(d+v)+(d+\frac{kI^*}{1+\alpha I^{*2}})\{(\varepsilon+d)(\gamma+d)+(\varepsilon+\gamma+2d)(d+v)\}+\frac{\varepsilon k^2I^*S^*}{(1+\alpha I^{*2})^2}-\frac{\varepsilon kS^*}{1+\alpha I^{*2}}[2d+v+\frac{kI^*}{1+\alpha I^{*2}}]$$

$$a_4=(d+\frac{kI^*}{1+\alpha I^{*2}})\{(\varepsilon+d)(\gamma+d)(d+v)\}+\varepsilon k(d+v)\times\frac{S^*}{1+\alpha I^{*2}}-\{\varepsilon k^2(d+v)\frac{I^*S^*}{(1+\alpha I^{*2})^2}+\frac{\gamma kvI^*}{1+\alpha I^{*2}}\}$$

here $a_1>0$ and $a_3>0$ if $\frac{\varepsilon kS^*}{1+\alpha I^{*2}}[2d+v+\frac{kI^*}{1+\alpha I^{*2}}]<0$ and $a_4>0$ if $\left\{\varepsilon k^2(d+v)\frac{I^*S^*}{(1+\alpha I^{*2})^2}+\frac{\gamma kvI^*}{1+\alpha I^{*2}}\right\}<0$, by direct calculation we have $a_1a_2a_3>a_3^2+a_1^2a_4$. Then by Routh-Hurwitz criteria, it follows that endemic equilibrium P^* is locally asymptotic stable.

V. Numerical Simulations

To see the dynamical behavior of system (1), solve the system by using the parameters:

Case I. $B=0.001$, $d=0.2$, $k=0.4$, $\alpha=0.1$, $\varepsilon=2.0$, $\gamma=0.2$, and $v=0.15$, then the basic reproduction number $R_0=0.004545454<1$ **(**figure-1).

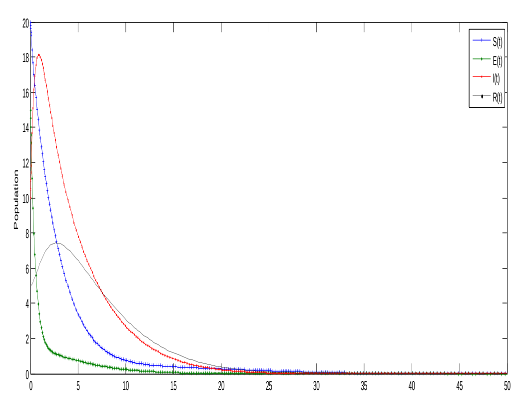

Figure-1

Case II. $B = 1.0$, $d = 0.04$, $k = 0.398$, $\alpha = 0.1$, $\varepsilon = 1.0$, $\gamma = 0.143$, and $\nu = 0.15$, then the basic reproduction number $R_0 = 52.28 > 1$ (figure-2).

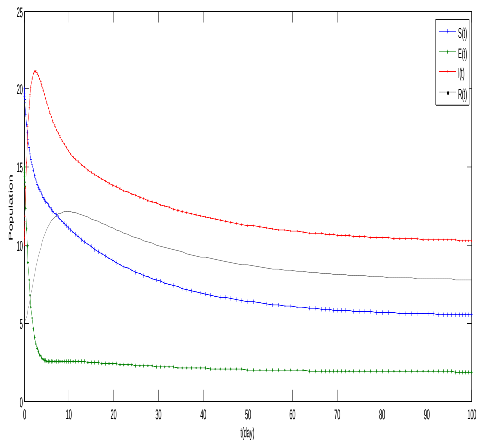

Figure-2

Case III. This figure shows that the dependence of I^* on the parameter α.

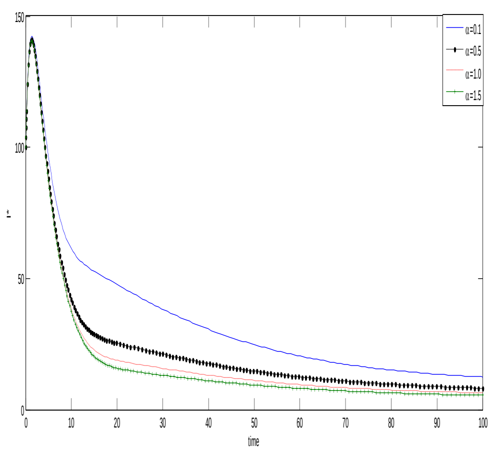

Figure-3

V. CONCLUSIONS

In this paper, we have carried out the stability of the equilibrium states using some of the tested parameters from the literature reviewed in this paper. In our model the basic reproduction number is less than unity, the disease-free equilibrium is locally asymptotically stable, and therefore, the disease dies out after some period of time. Similarly, when the basic reproduction number is greater than unity the disease is endemic. Lastly a numerical simulation provided when $R_0 < 1$ the disease-free equilibrium is stable (see figure 1), while $R_0 > 1$, the disease-free equilibrium unstable (see figure 2) and figure (3) shows that I^* decreses as α as increses.

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