A Study on Homogenous and Heterogenous Markov Processes

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Abstract
This Paper deals with the homogeneous and heterogeneous Markov chain. The application of it in the marketing of telecommunication product is discussed through homogeneous Markov models. Life time value of customer in interactive marketing is discussed in heterogeneous Markov model. Flexibility Markovian model plays a vital role in modeling the problems. Almost all the situations previously modelled are amenable to Markov chain modeling. The steady-state solution of the model is obtained iteratively. Some performance measures are derived. Finally, some important queueing models are derived as special cases of this model.

Markov chain: Let $\{X(t)$ be a Markov process which possesses Markov property and which takes only discrete values whether t is discrete or continuous. Then $\{X(t)$ is called as Markov chain. Mathematically, we define the Markov Chain as follows.

If P $\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots X_0 = a_0\}$

= P $\{X_n = a_n / X_{n-1} = a_{n-1}\}$ for all n.

Then the process X_n : n = 0, 1, 2, … are called as Markov Chain. Here a_1, a_2,,…a_n are called the states of the Markov Chain.

Homogeneous Markov Chain: If P_{ij}(n-1, n) = P_{ij}(m-1,m), then the Markov Chain is called the homogeneous Markov Chain or the chain is said to have stationary transition probabilities.

Steady - State distribution: If a homogeneous Markov chain is regular, then every sequence of state probability distributions will approach a unique fixed probability distribution called the stationary(state) or steady-state distribution of the Markov chain.

Mathematical Subject Classification: MSC 60K25, 90B22.

Key Words: homogeneous and heterogeneous Markov chain, Steady-state solution, Performance measures.

1. Introduction:

Stochastic point processes are commonly used to describe the time-to-failure phenomena in life data analysis and can be classified by the kind of failure time (lifetime) distribution and repair operation [3].

Various methods based on Markov models have been proposed in literature, including discrete time (Albert and Waclawiw, 1998) and continuous time models (Andersen et al., 1993).

Markov chain models had proved to be a useful tool in many fields, such as physics, chemistry, information sciences, economics, finances, mathematical biology, social sciences, and statistics for analyzing data.

Behavior of customers in their brand loyalty and those who change from brand to brand (switching) in marketing can be analyze only with the help of Markov analysis. In management, the application of Markov analysis will be fruitful only by applying mathematical background**.** This heterogeneity may be the result of unobservable factors that affect individuals' health.

A discrete time Markov chain, X_t for t = 1, 2 ,..., is a stochastic process with the Markov property, that is $p\{X_{t+1} = y / X_t = x, X_{t-1} = x_{t-1}, \dots\dots\dots\dots X_1 = x_1\} = p\{X_{t+1} = y / X_t = x\}$ for any states x, y.

Let $\{N(t), t \geq 0\}$ be a discrete Markov process. If the conditional probabilities $p\{N(t+s) = j / N(s) = i\}$ for all s, t ≥ 0, do not depend on s then the process is said to be time homogeneous.

The first model deals with application of homogeneous Markov model in telecommunication.

A first-order Markov process is based on the assumption that the probability of the next event (subscribers' choices of networks next month, in this case) depends upon the outcomes of the last event (subscribers' choice this month) and not at all on any earlier choices.

Now, we define the transition probability $p_{i,j}(t) = p\{N(t+s) = j / N(0) = i\}$ and the transition matrix P(t), whose element with index is $p_{i,j}(t)$. $p_{i,i}(0) = 1$and $p_{i,j}(0) = 0$ for $i \neq j$ so that $p(t) = 1$.

For a Markov process with time-homogeneous transition probabilities the so called Chapman-Kolmogorov equation implies $p_{i,j}(t+s) = \sum_{k=0}^{\infty} p_{i,k}(t) p_{k,j}(s)$.

This equation states that in order to move from state i to j in time (t+s), the queue size process N(t) moves to some intermediate state k in time t and then from k to j in the remaining time s. It also says how to compute the long –interval transition probability from a sum of short-interval transition probability components .An infinitesimal transition probability, denoted by $p_{i,j}(\Delta t)$, specify the immediate probabilistic behavior of a Markov process in that $\Delta t \to 0$.

Next model is behavior of the customer in terms heterogeneous Markovian.

Considering these aspects two models are proposed. The first model is an exponentially distributed random variable with constant parameters. In the second model the exponential time distribution in one of the systems is assumed to have an arbitrary distribution. In the first model the underlying stochastic process is identified as a Markov process and in the second model it is customer behavior of the stochastic. For both the models, the measures of system performance such that the steady-state solution of the model is obtained iteratively. Finally, some important queueing models are derived as special cases of these models.

In this paper, we analyze two stages of service subject to the application of marketing. The first stage is the marketing of telecommunication product in homogeneous Markov model and the second stage is the lifetime value of a customer which is an important and useful concept in interactive marketing , dealt with in case of heterogeneous Markov model.

The plan of the paper is as follows: In section 2, we analyze the model assumptions and in section 3, we provide the steady state analysis of the model. The measures of system performance are obtained in section 4.

2. Model description

This paper consists of two types of service provided to the customer. The type I is teleproduct service due to homogeneous of size k with probability c_k and the type II customer behavior of size k with probability d_k according to two Poisson processes with rate $\lambda_1 \bar{c} = \lambda_1 \sum_{k=1}^{\infty} k c_k$ and $\lambda_2 \bar{d} = \lambda_2 \sum_{k=1}^{\infty} k d_k$ respectively. Type II customers check the teleproduct of infinite capacity in order to seek service after random amount of time. All the customers in the service group behave independent of each other.

We assume the following to describe the queueing model of our study,

- Customer (Product) arrive at the system one by one in a Poisson process with arrival rate λ (>0).
- The availability of teleproduct and no stock in the system with probability are 1-α and α .
- As soon as the teleproduct of homogeneous model is finished, then the next to move is for the heterogeneous work in the customer, with probability $\beta_i (i = 1,2,3......M)$ or they may remain in the system to serve the next call if any, with probability β_0 where $\sum_{1=0}^{M} \beta_i = 1$.
- Each system undergoes three stages of service provided by a single server on a first come first out basis .The service times of the two stages follows different distribution with distribution function $\chi_i(v)$ and the density function.
- $\chi_i(v), i = 1,2$ .Assuming that they have finite moments $E(\chi_j^l)$ for $l \geq j$,j=1,2.
- The customers are served according to the first come, first serve rule.
- Various stochastic process involved in the system are independent of each other.

3. Definition and Equations governing the system

We first derive the system state equations for the queue size distribution at stationary point of time by treating three stages of service time as supplementary variables. Then we solve these equations to derive the probability generating function (PGF) for it.

The state of the system at time t can be described by the Markov process,

$$\{N(t), t \geq 0\} = \{X(t),\ \tau_1(t), \tau_2(t),\ \tau_3(t), t \geq 0\}$$

where X(t) denotes the system state 1,2,3 to be served being teleproduct homogeneous Markov, customer behavior in heterogeneous and homogeneous .

Assuming the system is in steady state condition and define S (t) as the queue size (excluding one service in service) at time t, $E_i^0(t)$ as the elapsed i^{th} stage of service for time t (1, 2, 3) respectively. We introduced the random variable X (t) as follows,

$$X(t)= \begin{cases} 1,\ \textit{the system is teleproduct due to homogeneous service} \\ 2,\ \textit{the system is in service customer behavior in hetrogeneous service} \\ 3,\ \textit{the system is in service customer behavior in homogeneous service} \end{cases}$$

Let N(t) be the system size of the server at time t. We introduced the supplementary variable

$$w(t) = \begin{cases} \tau_1(t),\ \ if\ C(t) = 1 \\ \tau_2(t), if\ C(t) = 2 \\ \tau_3(t), if\ \ C(t) = 3 \end{cases}$$

Where $\tau_1(t)$ is a teleproduct due to homogeneous Markov Model, $\tau_2(t)$ customer behavior due to heterogeneous Markov Model, $\tau_3(t)$ is a customer find and fix it homogeneous Markov Model.

The second chance to serve the customers in exponential distribution with parameter ξ .

The process $\{N(t), X(t), w(t), t \geq 0\}$ is a continuous time Markov process. We define the probabilities,

$$P_{0,0}(t) = P\{X(t) = 0, N(t) = 0\},$$
$$P_{1,n}(t)dx = P\{X(t) = 1, N(t) = n, x \leq \tau_1(t) < x + dx\}, x \geq 0, n \geq 1,$$
$$P_{2,n}(t)dx = P\{X(t) = 2, N(t) = n, x \leq \tau_2(t) < x + dx\}, x \geq 0, n \geq 0,$$
$$P_{3,n}^k(t)dx = P\{X(t) = 3, N(t) = n, x \leq \tau_3(t) < x + dx\}, x \geq 0, n \geq 0, k = 1,2,...M$$

4. Steady state distribution:

The queueing model is then, performed with the help of the following differential –difference equation:

1. $P_{0,t}(n)$ is the probability that there are n(n=0,1,2,…) teleproduct in the system due to homogeneous Markov model.
2. $P_{0,C1}(n)$ is the probability that there are n(n=0,1,2,…) customer behavior due to heterogeneous.
3. $P_{0,C2}(n)$ is the probability that there are n(n=0,1,2,…) customer behavior due to homogeneous.
4. $P_2(t) = P_{C1,C2}(t)$ and $P_{ci}(t) = P_{C1,0}(t) + P_{0,C2}(t)$

The steady state equation governing the model are given as,

$$\frac{d}{dt}P_{00}(t) = -\lambda P_{00}(t) + \mu_1 P_{C1,0}(t) + \mu_2 P_{C2,0}(t) \quad \text{……………………………...….(1)}$$

$$\frac{d}{dt}P_{C1,0}(t) = -(\lambda + \mu_1)P_{C1,0}(t) + \mu_2 P_{C1,0}(t) + \mu_2 P_{C1,C2}(t) + \lambda P_{0,0}(t) \quad \text{……......(2)}$$

$$\frac{d}{dt}P_{0,C2}(t) = -(\lambda+\mu_2)P_{,0,C2}(t) + \mu_1 P_{C1,C2}(t) + \lambda P_{0,0}(t) \quad \text{.......................(3)}$$

$$\frac{d}{dt}P_2(t) = -(\lambda+\mu_1+\mu_2)P_2(t) + (\mu_1+\mu_1+\xi p)P_3(t) + \lambda P_1(t) \quad \text{................(4)}$$

$$\frac{d}{dt}P_n(t) = -[(\frac{\lambda}{(n-2)+1}) + \mu_1 + \mu_2 + (n-2)\xi p)]P_n(t) + [(\mu_1 + \mu_1 + \{(n+1)-2\}\xi p)P_{n+1}(t)$$
$$+\left(\frac{\lambda}{(N-3)+1}\right)P_{n-1}(t);\ 3 \le n \le N-1 \quad \text{..................................(5)}$$

$$\frac{d}{dt}P_n(t) = -[(\frac{\lambda}{(n-2)+1})]P_{N-1}(t) - [(\mu_1 + \mu_1 + \{(N-2)\}\xi p)P_N(t) \quad \text{........(6)}$$ In steady state , $\lim_{n\to\infty} P_n(t) = P_n$, therefore, the steady state equations corresponding to equations (1)-(6) are as follows;

$$0 = -\lambda P_{00}(t) + \mu_1 P_{C1,0}(t) + \mu_2 P_{C2,0}(t) \quad \text{...(7)}$$

$$0 = -(\lambda+\mu_1)P_{C1,0}(t) + \mu_2 P_{C1,0}(t) + \mu_2 P_{C1,C2}(t) + \lambda P_{0,0}(t) \quad \text{......................(8)}$$

$$0 = -(\lambda+\mu_2)P_{,0,C2}(t) + \mu_1 P_{C1,C2}(t) + \lambda P_{0,0}(t) \quad \text{.....................................(9)}$$

$$0 = -(\lambda+\mu_1+\mu_2)P_2(t) + (\mu_1+\mu_1+\xi p)P_3(t) + \lambda P_1(t) \quad \text{.........................(10)}$$

$$0 = -[(\frac{\lambda}{(n-2)+1}) + \mu_1 + \mu_2 + (n-2)\xi p)]P_n(t) + [(\mu_1 + \mu_1 + \{(n+1)-2\}\xi p)P_{n+1}(t)$$
$$+\left(\frac{\lambda}{(N-3)+1}\right)P_{n-1}(t);\ 3 \le n \le N-1 \quad \text{..............................(11)}$$

$$0 = -[(\frac{\lambda}{(n-2)+1})]P_{N-1}(t) - [(\mu_1 + \mu_1 + \{(N-2)\}\xi p)]P_N(t) \quad \text{.............(12)}$$

On solving equations (7)-(9), we have

$$P_{C1,0} = \frac{\lambda+(\mu_1+\mu_2)\lambda}{2\lambda+\mu_1+\mu_2}\frac{\lambda}{\mu_1}P_{00} \quad \text{..(13)}$$

$$P_{0,C2} = \frac{\lambda+(\mu_1+\mu_2)\lambda}{2\lambda+\mu_1+\mu_2}\frac{\lambda}{\mu_2}P_{00} \quad \text{..(14)}$$

Addding (13) and (14),we get

$$P_1 = \frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda(\mu_1+\mu_2)}{\mu_1\mu_2}P_{00} \quad \text{..(15)}$$

Solving recursively equations (10)-(12),we get

$$P_n = \frac{1}{\{(n-2)+1\}!}\prod_{k=2}^{n}\frac{\lambda}{[\mu_1+\mu_2+(k-2)\xi p}\frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda(\mu_1+\mu_2)}{\mu_1\mu_2}P_{00}, 3 \le n \le N-1$$
$$\text{..(16)}$$

Using the normalization condition, $\sum_{n=0}^{N} P_n = 1$, we get

$$P_{00} = \left\{1 + \frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda(\mu_1+\mu_2)}{\mu_1\mu_2} + \sum_{n=2}^{N}\frac{1}{\{(n-2)+1\}!}\prod_{k=2}^{n}\frac{\lambda}{[\mu_1+\mu_2+(k-2)\xi p}P_{00}, 3 \le n \le N-1\right\}^{-1}$$
$$\text{..(17)}$$

Hence, the steady state probabilities of the system size are derived explicitly.

5. The measures of system performance:

In this section some important measures are derived. These can be used for the study of system performance,

(i) The length of the system

$$L_s = \frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda(\mu_1+\mu_2)}{\mu_1\mu_2}P_{00}+$$
$$\sum_{n=2}^{N}\frac{1}{\{(n-2)+1\}!}\prod_{k=2}^{n}\frac{\lambda}{[\mu_1+\mu_2+(k-2)\xi p}1+\frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda(\mu_1+\mu_2)}{\mu_1\mu_2}P_{00}$$

(ii) The length of the customers served

$$L_q = \mu_1\left\{\frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda}{\mu_1}\right\}P_{00}+\mu_2\left\{\frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda}{\mu_2}\right\}P_{00}$$
$$+(\mu_1+\mu_2)\sum_{n=2}^{N}\frac{1}{\{(n-2)+1\}!}\prod_{k=2}^{n}\frac{\lambda}{[\mu_1+\mu_2+(k-2)\xi p}\frac{\lambda+(\mu_1+\mu_2)}{2\lambda+\mu_1+\mu_2}\frac{\lambda(\mu_1+\mu_2)}{\mu_1\mu_2}P_{00}$$

(iii) Blocking probability

$$==1-\{P_{00}+P_0(1\}$$

(iv) Expected waiting time in the system

$$W_s = \frac{L_s}{\lambda\alpha}$$

6. Conclusion:

This paper is about the study of two models of Markovian with homogeneous and heterogeneous service time of a company and the customers. The stationary of the system size are derived .This paper analyzed the limit of finite capacity transient state to get time independent results. This same idea extended to the retrial queueing models.

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