Primality of L - Fuzzy Almost Ideals

P.Veerammal[#1], G.Velammal[*2]
Department of Mathematics, Saraswathi Narayanan College, Madurai, 625 022, Tamil Nadu, India
Department of Mathematics, Sri Meenakshi Government Arts College, Madurai, 625 002, Tamil Nadu, India

Abstract - *We consider L-Fuzzy Sets where the membership values do not necessarily form a chain. In our previous paper, the concept of L-Fuzzy Almost Ideal was introduced where L is not necessarily distributive. In this paper, we define and discuss the concept of primality in LFAI .*

Keywords **-** *L-Fuzzy Sets, Fuzzy Ideals, L-Fuzzy Ideals, Non-Distributive lattice, Level set, Almost Level Set, L-Fuzzy Almost Ideals, Prime L-fuzzy Almost Ideal, Semiprime L-Fuzzy Almost Ideal and Primary L-Fuzzy Almost Ideal.*

I. INTRODUCTION

Soon after the inception of concept of fuzzy sets [11] and L-Fuzzy sets [1] interest arose in fuzzy and L-fuzzy algebraic structures. Fuzzy subgroups and fuzzy ideals were studied [10]. The concept of primeness or primality was extended to fuzzy and L-fuzzy ideals ([2],[7],[8],[11], [12], [5], [6]). F.J.Lobillo, O.Cortadellas and G.Navarro [4] have given an excellent and extensive review of the several definitions of prime fuzzy ideal and semiprime fuzzy ideal and extended the concepts to non commutative rings.

In our previous paper [14], the concept of L-Fuzzy Almost Ideal was introduced.So far L-Fuzzy Ideals have been studied in the case where L is a distributive lattice. L- Fuzzy Almost Ideal is a generalisation of L-fuzzy ideal and L-Fuzzy Almost Ideal is defined even when the lattice L-is not necessarily distributive.

The aim of this current paper is to extend the concept of primality to L-Fuzzy Almost Ideal.

II. PRELIMINARIES

Let X be a nonempty subset, $(L, \leq, \vee, \wedge)$ be a complete distributive lattice, which has least and greatest elements, say 0 and 1 respectively. Relevant definitions are recalled in this section.

Definition 2.1 Let X be a nonempty set. A mapping $\mu: X\rightarrow [0,1]$ is called a fuzzy subset of X.

Definition 2.2 Let X be a nonempty set. A mapping $\mu: X\rightarrow L$ is called a L-fuzzy subset of X.

Definition 2.3 A fuzzy subset μ of a ring R is called a fuzzy ideal of R if for all x, y ∈ R,

$\mu(x-y) \geq \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq \mu(x) \vee \mu(y)$.

Definition 2.4 let μ be any fuzzy subset of a set X, t ∈ [0,1]. Then the set

$\mu_t = \{x \in X / \mu(x) \geq t\}$ is called a level set of μ. More generally if μ is L- fuzzy set defined by $\mu: X\rightarrow L$ then the set

$\mu_t = \{x\in X /\mu(x) \geq t\}$ is called a level set of μ.

Definition 2.5 If A, B are L-fuzzy sets their product is defined as follows;

$$A \circ B(x) = \bigvee_{\{(y,z)|x=yz\}} A(y) \wedge B(z)$$

Note Several definitions have been given for fuzzy prime ideals.To distinguish between the various definitions we adopt the notation suggested by F.J.Lobillo, O.Cortadellas and G.Navarro[4].

Definition 2.6 (D_0- prime) A non-constant fuzzy ideal $P: R\rightarrow [0,1]$ is said to be prime if, whenever $x_t\, y_s \leq P$ for any singletons x_t and y_s ,then $x_t \leq P$ or $y_s \leq P$.

Definition 2.7 (D_1- prime) A non-constant fuzzy ideal $P: R\rightarrow [0,1]$ is said to be prime if, whenever I o J ≤ P for some fuzzy ideal I and J, I ≤ P or J≤ P.

Definition 2.8 (D_2- prime) A non-constant fuzzy ideal $P: R\rightarrow [0,1]$ is said to be prime if the level ideal P_α is prime for any $P(0) \geq \alpha > P(1)$.

Definition 2.9 (D_3- prime) A non-constant fuzzy ideal $P: R\rightarrow [0,1]$ is said to be prime if, for any x,y∈R, whenever P(xy) = P(0),then P(x) = P(0) or P(y) = P(0).

Definition 2.10 (D_4- prime) A non-constant fuzzy ideal $P:R\rightarrow [0,1]$ is said to be prime if, for any x,y ∈ R, whenever P(xy) = P(x) or P(xy) = P(y).

Note The relationship between the various definitions of primes in case of commutative rings is summarized in the figure-I.

Definition 2.11 A fuzzy ideal μ of a ring R is called a fuzzy primary ideal if for all x,y ∈ R, either $\mu(xy) = \mu(x^n)$ or else $\mu(xy) \leq \mu(y^m)$ for some $n,m\in Z_+$.

Definition 2.12 A fuzzy ideal μ of a ring R is called a fuzzy semiprime ideal if for all

$x \in R$, $\mu(x^n) = \mu(x)$ for all $n \in Z_+$.

Definition 2.13 A fuzzy ideal μ of a ring R is called a fuzzy semi primary ideal if for all x,y ∈ R, either $\mu(xy) \leq \mu(x^n)$ or else

$\mu(xy) \leq \mu(y^m)$ for some $n,m \in Z_+$.

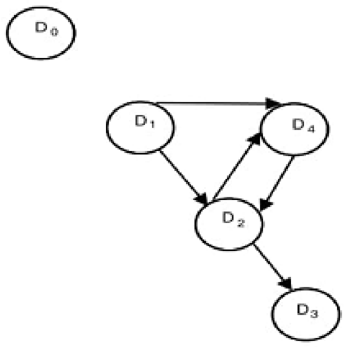
Definition 2.14 A L-fuzzy subset μ of a ring R is called a L-fuzzy ideal of R if,

(i)$\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for all x,y ∈ R,

(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all x,y ∈ R.

Figure- I

Relationship between the various definitions of fuzzy prime ideals in commutative rings



Definition 2.15 A L-fuzzy ideal μ of a ring R is called a L-fuzzy primary ideal, if for all x,y ∈ R, either μ(xy) = μ(x) or else μ(xy) ≤ μ(x^n) for some n ∈ Z_+.

Notation Consider μ: X→L. If L is totally ordered then for all x,y ∈ R, μ(x) and μ(y) are comparable. That is either μ(x) > μ(y) or μ(x) = μ(y) or μ(x) < μ(y). But if L is not totally ordered then are four possibilities

μ(x) > μ(y) or μ(x) = μ(y) or μ(x) < μ(y) or μ(x) and μ(y) are not comparable. We use the notation μ(x) ≮ μ(y) to mean , " μ(x) > μ(y) or μ(x) = μ(y) or μ(x) and μ(y) are not comparable".

Definition 2.16 Let R be a ring with unity. Let L be a lattice (L, ≤,∨,∧) not necessarily distributive with least and greatest element 0 and 1 respectively. μ: R→L with μ(0)=1 and μ(1)=0 is said to be L-fuzzy almost ideal if ,

(i) μ(x-y) ≮ μ(x) ∧ μ(y) for all x, y ∈ R.

(ii) μ(xy) ≮ μ(x) and μ(xy) ≮ μ(y) for all x,y ∈R.

Note If μ is a L-fuzzy almost ideal, then $\mu_1$is an ideal. But the other level sets need not be ideals.

III Prime L-fuzzy Almost Ideal

Definition 3.1 A non-constant L-fuzzy almost Ideal μ:R→L is said to be D_0 –Prime L-Fuzzy Almost Ideal if, whenever x_t y_s ≤ μ for any singletons x_t and y_s, then x_t ≤ μ or y_s ≤ μ.

Definition 3.2 A non-constant L-fuzzy almost Ideal μ:R→L is said to be D_1 –Prime L-Fuzzy Almost Ideal if, whenever I o J ≤ μ for some L-fuzzy almost Ideal I and J, I ≤ μ or J ≤ μ.

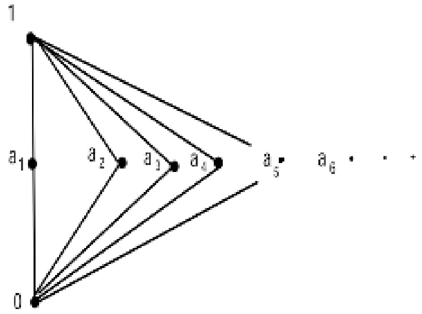
Definition 3.3 A non-constant L-fuzzy almost Ideal μ:R→L is said to be D_2 –Prime L-Fuzzy Almost Ideal if, the level set μ_α is prime ideal for any μ (0) ≥ α > μ (1).

Definition 3.4 A L-fuzzy almost Ideal μ:R→L is said to be D_3 –Prime L-Fuzzy Almost Ideal if it is non-constant and for all x,y∈R whenever μ(xy) = μ(0) then μ(x) = μ(0) or μ(y) = μ(0) .

Definition 3.5 Let μ: R→L be a L-fuzzy almost ideal is said to be D_4- Prime L –fuzzy almost ideal if for all x,y ∈ R,

μ(xy) = μ(x) or μ(y).

Example 3.6 An example of a D_3 –Prime L-Fuzzy Almost Ideal of the ring of integers.



Let R=Z. Let L be a lattice (L, ≤ ,∨, ∧,) defined by above Hasse diagram. Let p be a prime. Define μ: Z→L as follows.

$$\mu(x) = \begin{cases} 0 & if\ p \nmid x \\ a_i & if\ p^i | x, p^{i+1} \nmid x \\ 1 & if\ x = 0 \end{cases}$$

To show that μ is a L-fuzzy almost ideal it is enough to verify

Axiom 1: μ (x-y) ≮ μ(x) ∧ μ(y) for all x,y∈ R

Axiom2: μ(xy) ≮ μ(x) and μ(xy) ≮ μ(y) for all x,y ∈ R.

The Table-I summarises the verification.

So μ defines a L-Fuzzy Almost Ideal. Also μ(xy) = 1 only if xy = 0.Hence x or y = 0.Therefore μ(x) = 1 or μ(y) = 1.So μ defines a D_3 –Prime L-Fuzzy Almost Ideal.

Theorem 3.7 Any D_3 –Prime L-Fuzzy Ideal is an D_3 –Prime L-Fuzzy Almost Ideal.

The proof is obvious.

Theorem 3.8 Let μ: R → L be a L-fuzzy almost ideal. μ is D_3–Prime if and only if μ_1={x ∈ R| μ (x) = μ (0) = 1} is a prime ideal.

Proof: Assume μ: R → L is a D_3 –Prime L-fuzzy almost ideal. Suppose xy∈ $\mu_1$which implies that μ(xy) = 1. So μ(x) =1 or μ(y) =1 so x ∈ μ_1 or y ∈ μ_1.Thus μ_1 is a prime ideal.

Conversely, Suppose μ_1 is a prime ideal. Take μ(xy) = μ(0) = 1 which implies xy ∈ μ_1. Since μ_1 is a prime ideal either x ∈ μ_1 or y ∈ μ_1,then therefore μ(x) =1 or μ(y) =1.Hence μ is D_3 –Prime L-Fuzzy Almost Ideal.

Table - I
D_3 –Prime L-fuzzy Almost Ideal

μ(x)	μ(y)	μ(x-y)	μ(xy)	μ(x)∧μ(y)	Axiom 1	Axiom 2
0	a_i	0	a_i	0	$0 \nless 0$	$a_i \nless 0$ and $a_i \nless a_i$
a_i	a_j	$a_{min\,(i,j)}$	a_{i+j}	0	$a_{min\,(i,j)} \nless 0$	$a_{i+j} \nless_i$ and $a_{i+j} \nless a_j$
1	a_i	a_i	μ(0)=1	a_i	$a_i \nless a_i$	$1 \nless 1$ and $1 \nless a_i$

Table - II
Primary L-fuzzy Almost Ideal

$\mu(x)$	$\mu(y)$	$\mu(x-y)$	$\mu(xy)$	μ(x) ∧ μ(y)	Axiom 1	Axiom 2	Condition for primary LFAI
a_0	a_1	0	a_1	0	yes	yes	μ(xy) = μ(y)
a_0	0	0	0	0	yes	yes	μ(xy) = μ(x)
a_1	a_1 or a_2	a_1	a_2 or a_3	0	yes	yes	μ(xy) $\ngtr$ μ(y^3)
a_1	a_3	a_1	a_3	a_1	yes	yes	μ(xy) $\ngtr$ μ(y)
a_2	a_1 or a_2	a_1 or a_2	a_3	0 or a_2	yes	yes	μ(xy) $\ngtr$ μ(y^3)
a_2	a_1	a_1	a_3	a_1	yes	yes	μ(xy) = μ(x)
a_3	a_2	a_2	a_3	a_2	yes	yes	μ(xy) = μ(x)
a_3	a_3	a_3	a_3	a_3	yes	yes	μ(xy) = μ(x)

Theorem 3.9 Let R be a commutative ring with unity. If μ: R ⟶ L is a D_1- Prime L-fuzzy almost ideal then it is D_3 –Prime.

Proof: Suppose μ: R ⟶ L is a D_1- Prime L-fuzzy almost ideal and $\mu(xy) = 1$ .Let I and J be the characteristic maps of the ideals <x> and <y> respectively. It is clear $I \circ J \leq \mu$. Since μ is D_1- Prime either

$I \leq \mu$ or $J \leq \mu$.Say $I \leq \mu$.

Then $1 = I(x) \leq \mu(x)$.

So $\mu(x) = 1$. If $J \leq \mu$ one may similarly deduce that $\mu(y) = 1$.

Hence if $\mu(xy) = 1$ either $\mu(x) = 1$ or $\mu(y) = 1$.

So μ is D_3 –Prime.

Theorem 3.10 Let R be a commutative ring with unity. If μ: R ⟶ L is a D_4- Prime L-fuzzy almost ideal then it is D_3 –Prime.

Proof: Let μ is a L-fuzzy almost ideal. Let μ: R ⟶ L is a D_4- Prime L-fuzzy almost ideal which implies that μ (xy) = μ (x) or μ(y). Since μ_1 is a prime ideal iff μ is D_3- Prime [3.8]. Therefore μ (xy) = μ (0) = 1.

If μ(xy) = μ (x) or μ(y)

then μ (xy) = 1 ⇒ μ (x) = 1 or μ(y) = 1.

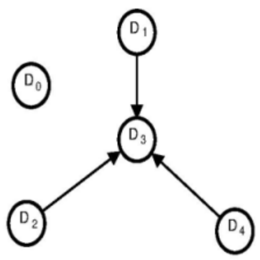
Hence μ is D_3 –Prime.

Theorem 3.11 Let R be a commutative ring with unity. If μ: R ⟶ L is a D_2- Prime L-fuzzy almost ideal then it is D_3 –Prime.

Proof: Let μ is a L-fuzzy almost ideal. Let μ: R ⟶ L is a D_2- Prime L-fuzzy almost ideal which implies that the level set μ_α is a prime L-fuzzy almost ideal for any μ(0) ≥ α > μ. Since μ_1 is a prime ideal iff μ is D_3- Prime [Theorem 3.8]. Hence μ is D_3 –Prime.

Note In the case of commutative ring the relation between the various definitions are summarized in the figure-II.

Figure- II
The relationship between the various definitions of primality of LFAI in commutative rings



Remark 3.12

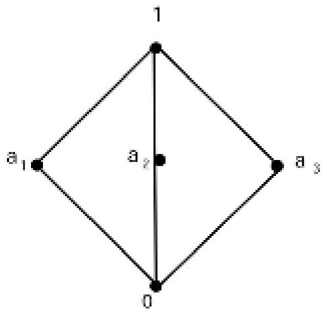
Mukerjee and Sen [5] have stated that " Let P be a non-null (i.e. $P_0 \neq (0)$) fuzzy prime ideal on Z. Then P has two distinct values. Conversely, if P be a fuzzy subset of Z such that P(n) = α_1 when p | n and P(n) = α_2 when p ∤ n ,where P is a fixed prime and $\alpha_1 > \alpha_2$, then p is a non-null fuzzy prime ideal on Z ".

` The example 3.6 shows this theorem is not true for D_3 –Prime L-Fuzzy Almost Ideal. Indeed in this case μ takes an infinite range of values 0,1,a_1, a_2, a_3, a_4,.........

IV *Semiprime L-fuzzy Almost Ideal*

Definition 4.1 A L-fuzzy almost ideal μ: R→L is said to be semiprime L –fuzzy almost ideal if for all x ∈ R, μ (x^n) ≯ μ(x) for all n ∈ Z_+.

Example 4.2 The following is an example of a Semiprime L-fuzzy Almost Ideal of the ring of integers.



Let R=Z. Let p,q be primes. Let (L, ≤ ,∨, ∧) be a lattice defined by above Hasse diagram. Note that L is not distributive.
Define μ: Z→L as follows.

$$\mu(x) = \begin{cases} 0 & if\ p \nmid x, q \nmid x \\ a_1 & if\ p | x, q \nmid x \\ a_2 & if\ pq | x \\ a_3 & if\ p \nmid x, q | x \\ 1 & if\ x = 0 \end{cases}$$

If μ(x) = 0 then $p \nmid x$, $q \nmid x$. Thus, it follows that $p \nmid x^n$, $q \nmid x^n$ ⇒ μ(x^n) = 0.
Hence μ (x^n) ≯ μ(x).
If μ(x)= a_1 then p| x , q ∤ x. Thus $p|x^n$, $q|x^n$ ⇒μ(x^n) = a_1. Hence μ(x^n)≯ μ(x).
If μ(x) = a_2 then pq| x . Thus $pq|x^n$ ⇒ μ(x^n) = a_2. Hence μ(x^n) ≯ μ(x).
If μ(x)=a_3 then p ∤ x , q|x .Thus $p \nmid x^n$, $q|x^n$ ⇒ μ(x^n)= a_3.Hence μ(x^n) ≯ μ(x).
If μ(x) = 1 then p| x , q|x .Thus $p|x^n$, $q|x^n$ ⇒μ(x^n) =1.Hence μ(x^n) ≯ μ(x). Thus we have μ is semiprime L-fuzzy Almost Ideal .

Theorem 4.3 Any semiprime L-fuzzy Ideal is a semiprime L-fuzzy Almost Ideal.
The proof is obvious.

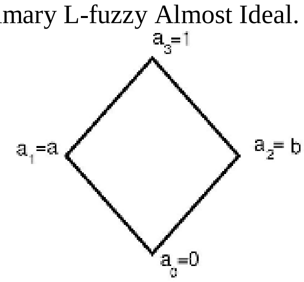
Theorem 4.4 Any D_4-Prime L-fuzzy Almost Ideal is a semiprime L-fuzzy Almost Ideal.

Proof: If μ: R→L is a D_4-Prime L-fuzzy Almost Ideal then μ(xy) = μ(x) or μ(y),for all x,y ∈ R. To prove that μ is semiprime L-fuzzy Almost Ideal, it is enough to prove that μ (x^n) ≯ μ(x), for all n ∈ Z_+.
Suppose μ is not semiprime L-fuzzy almost ideal, that is μ (x^n) ≯ μ(x) is not satisfied for some n, which implies that
μ (x^n) > μ(x), for some n ∈ Z_+ .... (1).
But μ (xx^{n-1}) = μ(x) or μ (x^{n-1}), because μ is D_4-Prime L-fuzzy almost ideal. Thus μ (x^n) = μ(x) or μ (x^n) = μ (x^{n-1}).
Case 1: If μ (x^n) = μ(x),
it is contradiction to (1).
Case 2: If μ(x^n) = μ(x^{n-1}).
By (1),we have μ(x^{n-1}) > μ(x)..........(2).
Then μ(x^{n-1}) = μ(xx^{n-2}) = μ(x) or μ(x^{n-2}).
If μ(x^{n-1}) = μ(x),it is contradiction to (2).
So μ(x^{n-1}) = μ(x^{n-2}).Proceeding like this, we get μ(x^n)=μ(x^{n-1}) = μ(x^{n-2}) = ...= μ(x^2)= μ(x) ⇒μ(x^n) = μ(x),which is a contradiction to(1).
It follows that μ(x^n) ≯ μ(x).Hence μ is semiprime L-fuzzy Almost Ideal.

V Primary L-fuzzy Almost Ideal

Definition 5.1 A L-fuzzy almost ideal defined by μ: R→L is a Primary L-fuzzy almost ideal if for all x,y ∈ R either μ(xy) = μ(x) or else μ(xy) ≯ μ(y^m) for some m ∈ Z_+.

Example 5.2 The following is an example of a Primary L-fuzzy Almost Ideal.



Let R = Z. Let (L, ≤,∨,∧) be defined by above Hasse diagram.
Define μ: Z→L as follows.

$$\mu(x) = \begin{cases} 0 & if\ p \nmid x \\ a_1 & if\ p | x, p^2 \nmid x \\ a_2 & if\ p^2 | x, p^3 \nmid x \\ 1 & if\ p^3 | x \end{cases}$$

The Table - II shows that both axioms for L-fuzzy almost ideal and also the condition for " primary " L-fuzzy almost ideal are satisfied.

Theorem 5.3 Any D_4-Prime L-fuzzy Almost Ideal is a primary L-fuzzy Almost Ideal.
The proof is obvious.

VI CONCLUSION

Swamy and Swamy [9] and many others have studied prime L-fuzzy ideals but under the assumption that L is distributive. We have extended the study to a more general case where L is not necessarily distributive. As noted in remark 4.12 there is a significant difference between the two cases.

In this paper we have extended the various definitions of prime L-fuzzy ideals to L-fuzzy almost ideals .The relationship between the different definitions has been investigated. The concepts of Semiprime L-fuzzy Almost Ideal and Primary L-fuzzy Almost Ideal have been introduced.

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