Some Equivalent Conditions on Quaternion -K-Normal Matrices

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ABSTRACT

The equivalent conditions of quaternion-𝓀-normal (q-𝓀-normal) and q-𝓀-unitary matrices are discussed also the q-𝓀-eigenvalues and the decomposition are given.

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***Key words**: q-𝓀-normal matrix, q-𝓀-unitary matrix, q-𝓀-eigenvalue and q-𝓀-eigenvector.*

1. INTRODUCTION

Let $A \in C_{n\times n}$, If $AA^* = A^*A$ then A is normal [2] K be the permutation matrix associated with the permutation 𝓀(x) on the set of positive integers S={1,2,3,……..,n}. A complex square matrix A is said to be 𝓀-symmetric real if $A = KA^TK$. If $A = KA^*K$ then 𝓀-Hermitian [4] with $K^2 = I$, $K = K^{-1} = K^* = K^T$. A matrix is a quaternion matrix if the elements of A belong to quaternion field H [5]. Some equivalent conditions were proved in [1]. In this paper, with the support of [1]. Some equivalent conditions are given for q-𝓀-normal matrices and q-𝓀-unitary matrices over quaternion field H [5] with q-𝓀-eigenvalues and decomposition.

2. DEFINITION AND THEOREM

Definition 2.1:[3] $A \in H_{n\times n}$ is said to be q-𝓀-normal if $AKA^*K = KA^*KA$.

Theorem 2.2:[1] Let $A = \left(a_{ij}\right)$ be an $n \times n$ square quaternion matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. The following statements are equivalent:

1. A is q-𝓀-normal, that is $AKA^*K = KA^*KA$.
2. A is q-𝓀-unitarily diagonalizable, namely there exists a q-𝓀-unitary matrix U of the same size such that $KU^*KAU =$ diag $\left(\lambda_1, \lambda_2, \dots, \lambda_n\right)$.
3. there exists a polynomial $p(x)$ such that $A^* = p(A)$.
4. there exists a set of q-𝓀-eigenvectors of A that form an orthonormal basis for H_n.
5. every q-𝓀-eigenvector of A is an q-𝓀-eigenvector of KA^*K.
6. $A = B + iC$ for some B and C q-𝓀-Hermitian, and $BC = CB$.
7. if U is a q-𝓀-unitary matrix such that $KU^*KAU = \begin{pmatrix} B & C \\ O & D \end{pmatrix}$ with B and D square, then B and D are q-𝓀-normal and $C = O$.

8. $tr\left(KA^{*}KA\right)^{2}=tr\left(\left(KA^{*}K\right)^{2}A^{2}\right)$.
9. $KA^{*}K=AU$ for some q-*k*-unitary U .
10. $KA^{*}K=VA$ for some q-*k*-unitary V .
11. A commutes with $A+KA^{*}K$.
12. A commutes with $A-KA^{*}K$.
13. $A+KA^{*}K$ and $A-KA^{*}K$ commute.
14. A commutes with $KA^{*}KA$.

Proof:

$(2)\Leftrightarrow(1)$**:** We show that $(1)\Rightarrow(2)$

Let $A=\left(KU^{*}K\right)TU$ be a Schur decomposition of A with $T\left(KT^{*}K\right)=\left(TKT^{*}K\right)T.$ It sufficies to show that the upper-triangular matrix T is diagonal.

$$A=\left(KU^{*}K\right)TU$$
$$A^{*}=U^{*}T^{*}\left(KUK\right)$$
$$KA^{*}K=K\left[U^{*}T^{*}KUK\right]K=KU^{*}T^{*}KUKK=KU^{*}T^{*}KU=KU^{*}KKT^{*}KU$$
$$=\left(KU^{*}K\right)\left(KT^{*}K\right)U$$

Now, $$A\left(KA^{*}K\right)=\left[KU^{*}KTU\right]\left(KU^{*}K\right)\left(KT^{*}K\right)U$$
$$=KU^{*}KTUKU^{*}KKT^{*}KU$$
$$=KU^{*}KT\left(UKU^{*}K\right)KT^{*}KU$$
$$=\left(KU^{*}K\right)T\left(KT^{*}K\right)U$$
$$\left(KA^{*}K\right)A=\left[\left(KU^{*}K\right)\left(KT^{*}K\right)U\right]\left[\left(KU^{*}K\right)TU\right]$$
$$=\left(KU^{*}K\right)\left(KT^{*}K\right)TU$$
$$=\left(KU^{*}K\right)T\left(KT^{*}K\right)U$$

Thus, $\left(KA^{*}K\right)A=A\left(KA^{*}K\right)$.

By comparing the diagonal entries on both sides of the latter identify, we have $t_{ij}=0.$ Whenever $i<j,$ Thus T is diagonal.

$(3)\Leftrightarrow(2)$**:** To show that (2) implies (3) we choose a polynomial $p(x)$ of degree at most $n-1,$ by interpolation, such that $p(\lambda_i)=\overline{\lambda_i},\ i=1,2,\ldots,n$

Thus if $A=(KU^{*}K)\,\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)U$ for some q-*k*-unitary matrix U , then

$$KA^{*}K=(KU^{*}K)\,\mathrm{diag}\left(\overline{\lambda_1},\overline{\lambda_2},\ldots,\overline{\lambda_n}\right)U$$

$$=(KU^{*}K)\,\mathrm{diag}\left(p(\lambda_1),p(\lambda_2),\ldots p(\lambda_n)\right)U$$
$$=(KU^{*}K)\,p\,(\mathrm{diag}\,(\lambda_1,\lambda_2,\ldots.\lambda_n))U$$
$$=p((KU^{*}K)\,\mathrm{diag}\,(\lambda_1,\lambda_2,\ldots\ldots.,\lambda_n)U)$$
$$=p(A)$$

For the other direction, if $KA^{*}K=p(A)$ for some polynomial then

$$\left(KA^{*}K\right)A=p(A)A=Ap(A)=A\left(KA^{*}K\right)$$

$(4)\Leftrightarrow(2)$: If $KU^{*}KAU=\mathrm{diag}\left(\lambda_1,\lambda_2,\ldots\lambda_n\right)$ then pre multiplying by U

$$AU=U\,\mathrm{diag}\left(\lambda_1,\lambda_2,\ldots\lambda_n\right)$$

$\Rightarrow Au_i=\lambda_i u_i, i=1,2,\ldots n$ where u_i is the i^{th} column vectors of U are q-𝓀-eigenvector of A. That is u_i is q-𝓀-eigenvector of A corresponding to λ_i for $i=1,2,\ldots n.$ And also they from an orthonormal basis for H_n, Since U is q-𝓀-unitary matrix.

Conversely, if A has a set of q-𝓀-eigenvectors that form an orthonormal basis for H_n, then the matrix U consisting of these vector as columns is q-𝓀-unitary and satisfies.

$$KU^{*}KAU=\mathrm{diag}\left(\lambda_1,\lambda_2,\ldots\lambda_n\right)$$

$(5)\Leftrightarrow(1)$:Assume that A is q-𝓀-normal and let u be an q-𝓀-eigenvector of A corresponding to q-𝓀-eigenvalue λ. Extend u to a q-𝓀-unitary matrix with u as the first column: then

$$KU^{*}KAU=\begin{pmatrix}\lambda & \alpha\\ O & A_1\end{pmatrix}$$

The q-𝓀-normality of A acts $\alpha=0$.

Taking the conjugate transpose and by a simple computation, u is an q-𝓀-eigenvector of $KA^{*}K$ corresponding to the q-𝓀-eigenvalue $\overline{\lambda}$ of $KA^{*}K.$

We use induction on n. Note that $Ax=\lambda x$

$$\Leftrightarrow\left(KU^{*}KAU\right)\left(KU^{*}K\right)x=\left(KU^{*}K\right)Ax$$
$$=\left(KU^{*}K\right)\lambda x$$
$$=\lambda\left(KU^{*}K\right)x$$

for any n-square q-𝓀-unitary matrix U. Thus when considering $Ax=\lambda x,$ we may assume that A is upper-triangular by Schur decomposition.

Taking $e_1=(1,0,\ldots,0)^T$. Then e_1 is an q-𝓀-eigenvector of A. Hence by assumption, $e_1=\overline{e_1}$ is an q-𝓀-eigenvector of $KA^{*}K$. A direct computation yields that the first column of $KA^{*}K$ must consist of zeros except the first component. Thus, if we write

$$A=\begin{pmatrix}\lambda_1 & O\\ \overline{O} & B\end{pmatrix}, \text{ then } KA^*K=\begin{pmatrix}\overline{\lambda_1} & O\\ O & K_1B^*K_1\end{pmatrix}$$

Since every q-𝓀-eigenvector of A is an q-𝓀-eigenvector of KA^*K, this property is inherited by B and $K_1B^*K_1$. An induction hypothesis on B shows that A is diagonal. This follows that A is q-𝓀-normal. $\left[(1)\Leftrightarrow(2)\right]$

$(6)\Leftrightarrow(1)$: We assume that $A=B+iC$ Where B and C are q-𝓀-Hermitian with $BC=CB$ then $KA^*K=KB^*K-iKC^*K$

It is enough to show that $A\left(KA^*K\right)=\left(KA^*K\right)A$

Now,

$$\begin{aligned}A\left(KA^*K\right)&=(B+iC)K(B+iC)^*K\\&=(B+iC)\left(KB^*K-iKC^*K\right)\\&=BKB^*K-iBKC^*K+iCKB^*K+CKC^*K\\&=\mathrm{KB^*KB-iKC^*KB+iKB^*KC+KC^*KC}\quad [\mathrm{BC=CB} \text{ and } \mathrm{C,B} \text{ are q-𝓀-Hermitian}]\\&=\left(KB^*K-iKC^*K\right)B+i\left(KB^*K-iKC^*K\right)C\\&=\left(KB^*K-iKC^*K\right)(B+iC)\\&=K(B+iC)^*K(B+iC)\\&=KA^*KA\\&=\left(KA^*K\right)A\end{aligned}$$

Thus A is q-𝓀-normal.

Conversely,

$$\text{Let } B=\frac{A+KA^*K}{2} \text{ and } C=\frac{A-KA^*K}{2i}$$

$$\begin{aligned}B+iC&=\frac{A+KA^*K}{2}+i\frac{A-KA^*K}{2i}\\&=\frac{2A}{2}\\&=A\end{aligned}$$

$$A=B+iC$$

$$KB^*K=K\left(\frac{A+KA^*K}{2}\right)^*K$$

$$= K\left(\frac{A^* + KAK}{2}\right)K$$

$$= \frac{KA^*K + A}{2}$$

$$= B$$

Thus B is q-*k*-Hermitian.

$$KC^*K = K\left(\frac{A - KA^*K}{2i}\right)^* K$$

$$= K\left(\frac{A^* - KA^*K}{2i}\right)K$$

$$= K - \left(\frac{A^* - KAK}{2i}\right)K$$

$$= -\left[\frac{KA^*K - A}{2i}\right]$$

$$= \frac{A - KA^*K}{2i}$$

$$= C$$

$(7) \Leftrightarrow (1)$: We show that $(1) \Rightarrow (7)$. The other direction is simple upon computation, we have $AKA^*K = KA^*KA$

$$A = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

$$A^* = \begin{bmatrix} B^* & O \\ C^* & D^* \end{bmatrix}$$

$$KA^*K = \begin{bmatrix} KB^*K & O \\ KC^*K & KD^*K \end{bmatrix}$$

$$\left(KA^*K\right)A = \begin{bmatrix} KB^*K & O \\ KC^*K & KD^*K \end{bmatrix}\begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

$$= \begin{bmatrix} KB^*KB & KB^*KC \\ KC^*KB & KC^*KC + KD^*KD \end{bmatrix}$$

$$A\left(KA^*K\right) = \begin{bmatrix} B & C \\ O & D \end{bmatrix}\begin{bmatrix} KB^*K & O \\ KC^*K & KD^*K \end{bmatrix}$$

$$=\begin{bmatrix} BKB^*K+CKC^*K & CKD^*K \\ DKC^*K & DKD^*K \end{bmatrix}$$

Thus,

$$KB^*KB=BKB^*K+CKC^*K \text{ and } KC^*KC+KD^*KD=DKD^*K$$

By taking the trace for both sides of the first identity and noticing that

$$tr\left(BKB^*K\right)=tr\left(KB^*KB\right), \text{ we obtain } tr\left(CKC^*K\right)=0$$

This forces $C=O$. Thus B is q-*k*-normal.

Similarly,

$$tr\left(DKD^*K\right)=tr\left(KD^*KD\right)$$

Thus

$$tr\left(KC^*KC\right)=0$$

$\Rightarrow D$ is also q-*k*-normal and $C=O$

So $KU^*KAU=\begin{bmatrix} B & C \\ O & D \end{bmatrix}$ for B and D q-*k*-normal and $C=O$.

Conversely it is easy to prove $(7)\Rightarrow(1)$ by using $(2)\Rightarrow(1)$

$(9)\Leftrightarrow(1)$: If $A^*=KAUK, A=K(AU)^*K$ for some q-*k*-unitary then

$$\left(KA^*K\right)A=K\left(KAUK\right)K\left[K\left(AU\right)^*K\right]$$
$$=\left(AU\right)K\left(AU\right)^*K$$
$$=\left(AU\right)KU^*A^*K$$
$$=AU\left(KU^*K\right)\left(KA^*K\right)$$
$$=A\left(KA^*K\right)$$
$$\left(KA^*K\right)A=A\left(KA^*K\right)$$

Therefore A is q-*k*-normal.

For the converse, we show $(2)\Rightarrow(9)$.

Let $A=(KV^*K)$ diag $\left(\lambda_1,\lambda_2,\ldots,\lambda_n\right)V$, where V is q-*k*-unitary.

Take $KU^*KU=(KV^*K)$ diag $\left(l_1,l_2,\ldots,l_n\right)V$, where $l_i=\dfrac{\overline{\lambda_i}}{\lambda_i}$ if $\lambda_1\neq 0$ and $l_i=1$ otherwise for $i=1,2,\ldots,n$ then

$$KA^*K=(KV^*K)\text{ diag }\left(\overline{\lambda_1},\overline{\lambda_2},\ldots,\overline{\lambda_n}\right)V$$
$$=(KV^*K)\left(\lambda_1,\lambda_2,\ldots,\lambda_n\right)V\ (KV^*K)\text{ diag }\left(l_1,l_2,\ldots,l_n\right)V$$
$$=A\left(KU^*KU\right)$$

Similarly, (10) is equivalent to (1).

$(11)\Rightarrow(1)$: A commutes with $A+KA^{*}K$

$$A\left(A+KA^{*}K\right)=\left(A+KA^{*}K\right)A$$
$$A^{2}+AKA^{*}K=A^{2}+KA^{*}KA$$
$$AKA^{*}K=A^{2}+KA^{*}KA-A^{2}$$
$$AKA^{*}K=KA^{*}KA$$

Therefore A is q-*k*-normal.

$(12)\Rightarrow(1)$: A commutes with $A-KA^{*}K$

$$A\left(A-KA^{*}K\right)=\left(A-KA^{*}K\right)A$$
$$A^{2}-AKA^{*}K=A^{2}-KA^{*}KA$$
$$-AKA^{*}K=-KA^{*}KA$$
$$AKA^{*}K=KA^{*}KA$$

Therefore A is q-*k*-normal.

$(13)\Rightarrow(1)$: $A+KA^{*}K$ and $A-KA^{*}K$ commute

$$\left(A+KA^{*}K\right)\left(A-KA^{*}K\right)=\left(A-KA^{*}K\right)\left(A+KA^{*}K\right)$$
$$A^{2}-AKA^{*}K+KA^{*}KA-\left(A^{*}\right)^{2}=A^{2}+AKA^{*}K-KA^{*}KA-\left(A^{*}\right)^{2}$$
$$-AKA^{*}K-KA^{*}KA=AKA^{*}K-KA^{*}KA$$
$$-AKA^{*}K-AKA^{*}K=-KA^{*}KA-KA^{*}KA$$
$$-2A\left(KA^{*}K\right)=-2\left(KA^{*}K\right)A$$
$$AKA^{*}K=KA^{*}KA$$

Therefore A is q-*k*-normal.

$(14)\Rightarrow(8)$: If A commutes with $KA^{*}KA$ then

$$A\left(KA^{*}KA\right)=\left(KA^{*}KA\right)A$$
$$AKA^{*}KA=KA^{*}K\left(A^{2}\right)$$

Multiply both sides by $KA^{*}K$ from the left to get

$$\left(KA^{*}K\right)A\left(KA^{*}KA\right)=\left(KA^{*}K\right)\left(KA^{*}K\right)\left(A^{2}\right)$$
$$\left(KA^{*}KA\right)^{2}=\left(KA^{*}K\right)^{2}\left(A^{2}\right)$$

Take the trace of both sides $tr\left(KA^{*}KA\right)^{2}=tr\left[\left(KA^{*}K\right)^{2}\left(A^{2}\right)\right]$.

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