Some cycle and path related strongly*-graphs

I. I. Jadav[1], G. V. Ghodasara[2]

[1] Research Scholar, R. K. University, Rajkot−360020, India.

[2] H. & H. B. Kotak Institute of Science,Rajkot−360001, India.

***Abstact* —A graph with n vertices is said to be strongly*-graph if its vertices can be assigned the values $\{1, 2, \ldots, n\}$ in such a way that when an edge whose end vertices are labeled i and j, is labeled with the value $i+j+ij$ such that all edges have distinct labels. Here we derive different strongly*-graphs in context of some graph operations.**

Key words : Strongly*-labeling, Strongly*-graph.
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I. Introduction

By a graph G, we mean a simple, finite, undirected graph.

Definition I.1. *[1] A graph G with n vertices is said to be strongly*-graph if there is a bijection $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ such that the induced edge function $f^* : E(G) \longrightarrow \mathbb{N}$ defined as $f^*(e = uv) = f(u) + f(v) + f(u) \cdot f(v)$ is injective. Here f is called strongly*-labeling of graph G.*

Definition I.2. *[2] A chord of cycle C_n, $n \geq 4$, is an edge joining two non-adjacent vertices of C_n.*

Definition I.3. *[1] Two chords of cycle C_n, $n \geq 5$, are said to be twin chords if they form a triangle with an edge of C_n.*

For positive integers n and p with $3 \leq p \leq (n-2)$, $C_{n,p}$ is the graph consisting of a cycle C_n with twin chords where the chords form the cycles C_p, C_3 and C_{n+1-p} without chords with the edges of C_n.

Definition I.4. *[1] The cycle with triangle is a cycle with three chords which by themselves form a triangle.*

For positive integers p, q, r and $n \geq 6$ with $p + q + r + 3 = n$, $C_n(p, q, r)$ denotes the cycle with triangle whose edges form the edges of cycles C_{p+2}, C_{q+2} and C_{r+2} without chords.

Definition I.5. *[1] The crown $(C_n \bigodot K_1)$ is obtained by joining a pendant vertex to each vertex of cycle C_n by an edge.*

Definition I.6. *[1] Tadpole, $T(l, r)$ is the graph in which path of length r is attached to any one vertex of cycle C_l by a bridge. $T(l, r)$ has $l + r$ vertices and $l + r$ edges.*

Definition I.7. *[1] The middle graph of a graph G (with two or more vertices), denoted by $M(G)$, is the graph whose vertex set is $V(G) \cup E(G)$ in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.*

Definition I.8. *[1] The total graph of a graph G (with two or more vertices), denoted by $T(G)$, is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are adjacent vertices or adjacent edges in G or one is a vertex of G and the other is an edge incident on it.*

Definition I.9. *[1] The split graph of a graph G, denoted by $spl(G)$, is the graph obtained by adding a new vertex v' to each vertex v such that v' is adjacent to every vertex that is adjacent to v in G.*

Definition I.10. *[1]The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G' and G'' of graph G. Join each vertex u' of G' to the neighbours of the corresponding vertex u'' of G''.*

Adiga and Somashekara[3] proved that all trees, cycles and grids are strongly*-graphs. In the same paper they considered the problem of determining the maximum number of edges in any strongly*-graph of given order and relates it to the corresponding problem for strongly multiplicative graphs.
For all others standard terminology and notations we follow Harary[2].

II. Main Results

Theorem II.1. *Cycle C_n with one chord is strongly*-graph for all $n \in \mathbb{N}$, where chord forms a triangle with two edges of C_n.*

Proof. Let G be the cycle C_n with one chord. Let $\{v_1, v_2, \ldots, v_n\}$ denote the successive vertices of C_n, where v_1 is adjacent to v_n and v_i is adjacent to v_{i+1}, $1 \leq i \leq n-1$. Let $e = v_1v_3$ be the chord of C_n. Note that $d(v_1) = d(v_3) = 3$ and $d(v_i) = 2$, $2 \leq i \leq n$, $i \neq 3$.
We define vertex labeling function $f : V(G) \to \{1, 2, \ldots, n\}$ as follows.

$$f(v_i) = \begin{cases} 2i-1; 1 \leq i \leq \lceil \frac{n}{2} \rceil. \\ 2(n-i+1); (\lceil \frac{n}{2} \rceil + 1) \leq i \leq n. \end{cases}$$

Hence the vertex labels on one half of C_n are monotonically increasing (consecutive) odd numbers, whereas vertex labels on the other half of C_n are monotonically increasing (consecutive) even numbers. Hence the produced edge labels must be different. Further the chord has $(1, 5) = 11$ label which is unique with respect to labeling f of the graph. Hence above defined labeling pattern satisfies the conditions of strongly*-labeling. i.e. Cycle C_n with one chord is strongly*-graph. □

Example 1. *Strongly*-labeling of cycle C_8 with one chord is shown in Figure 1.*

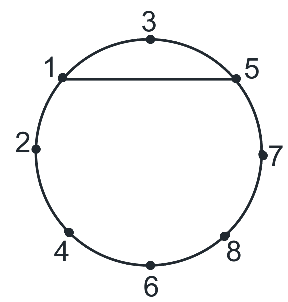

Fig. 1

Corollary 1. *Cycle C_n with twin chords $C_{n,3}$ is strongly*-graph for all $n \in \mathbb{N}$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ denote the successive vertices of the $C_{n,3}$ where v_1 is adjacent to v_n and v_i is adjacent to v_{i+1}, $1 \leq i \leq n-1$. Let $e_1 = v_1v_3$ and $e_2 = v_1v_5$ be two chords of $C_{n,3}$. Note that $d(v_1) = 4$, $d(v_3) = d(v_5) = 3$ and $d(v_i) = 2$, $2 \leq i \leq n$, $i \neq 3, i \neq 5$. We define vertex labeling function $f : V(C_{n,3}) \to \{1, 2, \ldots, n\}$ same as per the labeling defined in *Theorem 1*.
Here vertex labels on one half of the $C_{n,3}$ are monotonically increasing (consecutive) odd numbers, whereas vertex labels on the other half of the $C_{n,3}$ are monotonically increasing (consecutive) even numbers. Hence the produced edge labels must be different. Further the vertex with label 1 is adjacent to the vertices with label 5, 3 and 8 which are unique with respect to the labeling f of the given graph. Hence above defined labeling pattern satisfies the

conditions of strongly*-labeling. i.e. $C_{n,3}$ with twin chords is strongly*-graph. □

Example 2. *Strongly*-labeling of cycle* C_8 *with twin chords is shown in Figure 2.*

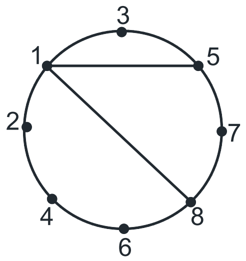

Fig. 2

Corollary 2. *Cycle with triangle* $C_n(1, 1, n-5)$ *is strongly*-graph for all* $n \in \mathbb{N}$.

Proof. Let G be the cycle with triangle $C_n(1, 1, n-5)$. Let $\{v_1, v_2, \ldots, v_n\}$ denote the successive vertices of the G such that v_1 is adjacent to v_n and v_i is adjacent to v_{i+1}, $1 \leq i \leq n-1$. Let $e_1 = v_1v_3$, $e_2 = v_1v_5$ and $e_3 = v_3v_5$ be three chords of C_n. Note that $d(v_1) = d(v_3) = d(v_5) = 4$ and $d(v_i) = 2$, $2 \leq i \leq n$, $i \neq 3, i \neq 5$.
We define vertex labeling function $f : V(G) \to \{1, 2, \ldots, n\}$ same as per the labeling defined in *Theorem 1*.
Here vertex labels on one half of the $C_n(1, 1, n-5)$ are monotonically increasing (consecutive) odd numbers, whereas vertex labels on the other half of the $C_n(1, 1, n-5)$ are monotonically increasing (consecutive) even numbers. Hence the produced edge labels must be different. Further the chords have $(1, 5) = 11$, $(1, 6) = 13$ and $(5, 6) = 41$ labels which are unique with respect to labeling f of the graph G. Hence above defined labeling pattern satisfies the conditions of strongly*-labeling. i.e. C_n with triangle is strongly*-graph. □

Example 3. *Strongly*-labeling of cycle* C_7 *with triangle is strongly*-graph shown in Figure 3.*

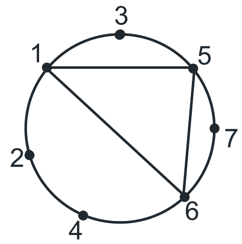

Fig. 3

Theorem II.2. *The crown* $C_n \bigodot K_1$ *is strongly*-graph for all* $n \in \mathbb{N}$.

Proof. Let $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$ be the vertices of the crown $C_n \bigodot K_1$, where $\{v_1, v_2, \ldots, v_n\}$ are the vertices corresponding to cycle C_n and $\{v'_1, v'_2, \ldots, v'_n\}$ are the pendant vertices. Here v'_i is adjacent to v_i, $i = 1, 2, \ldots, n$.
To define vertex labeling function $f : V(C_n \bigodot K_1) \to \{1, 2, 3, \ldots, 2n\}$ we consider the following cases.
Case 1: n is odd.
$f(v_i) = 2i$; $1 \leq i \leq n$.
$f(v'_i) = 2i - 1$; $1 \leq i \leq n$.
When n is odd, the labels for the vertices v_i are consecutive odd numbers, whereas the labels for the vertices v'_i are consecutive even numbers. Therefore the labels produced for the edges $v_iv'_i$ are odd and in increasing order. Further the edges v_iv_{i+1}, $1 \leq i \leq n$ are labeled with even labels in increasing order. Hence all edges will have different labels.
Case 2: n is even.

$$f(v_i) = \begin{cases} 4i - 1; 1 \leq i \leq \frac{n}{2}. \\ 4(n - i + 1); (\frac{n}{2} + 1) \leq i \leq n. \end{cases}$$

$$f(v'_i) = \begin{cases} 4i - 3; 1 \leq i \leq \frac{n}{2}. \\ 4(n - i) + 2; (\frac{n}{2} + 1) \leq i \leq n. \end{cases}$$

When n is even, for any two edge labels produced in the graph,
$\{(4j-3) + (4j-1) + (4j-3) \cdot (4j-1)\} \neq$

$\{(4j-1)+(4j+3)+(4j-1)\cdot(4j+3)\} \neq \{2n+(2n-1)+2n\cdot(2n-1)\} \neq \{4j+(4j-2)+4j\cdot(4j-2)\} \neq \{4j+4(j+1)+4j\cdot 4(j+1)\} \neq 19$, $1 \leq j \leq \frac{n}{2}$.

Hence above defined labeling pattern satisfies the conditions of strongly*-labeling. i.e. Crown $C_n \odot K_1$ is strongly*-graph for all n. □

Example 4. *Strongly*-labeling of crown* $C_6 \odot K_1$ *is shown in Figure 4.*

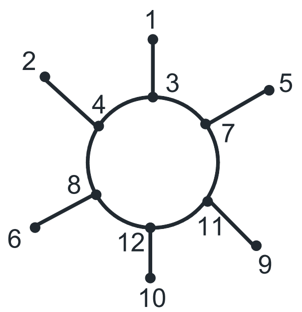

Fig. 4

Theorem II.3. *Tadpoles* $T(l, r)$ *are strongly*-graph for all* $l, r \in \mathbb{N}$.

Proof. Let $\{v_1, v_2, \ldots, v_l\}$ be the vertices of the cycle C_l and $\{v_{l+1}, v_{l+2}, \ldots, v_{l+r}\}$ be the vertices of path P_r attached to vertex v_l of C_l in the $T(l, r)$. Let $e = v_l v_{l+1}$ be the bridge joining vertex v_l of C_l and vertex v_{l+1} of the P_r. Note that $|V(T(l,r))| = |E(T(l,r))| = l + r$.

To define vertex labeling function $f : V(T(l,r)) \to \{1, 2, 3, \ldots, l+r\}$, We consider the following cases.

Case 1: $l \neq 2 + \sum_{i=3}^{m} i$, where $m \in \mathbb{N} - \{1, 2\}$.

$f(v_i) = i$, $1 \leq i \leq l + r$.

Case 2: $l = 2 + \sum_{i=3}^{m} i$, where $m \in \mathbb{N} - \{1, 2\}$.

$f(v_l) = l + 1$, $f(v_{l+1}) = l$.

$f(v_i) = i$, $1 \leq i \leq l + r$, $i \neq l, l + 1$.

Since the labeling f defined above is strictly increasing, for any two edges with end vertices labeled by vertex labels (i, j) and (s, t) respectively, $(i \neq j \neq s \neq t)$ we have $(i+j+ij) \neq (s+t+st)$. Hence above defined labeling pattern satisfies the conditions of strongly*-labeling. i.e. $T(l, r)$ is strongly*-graph. □

Example 5. *Strongly*-labeling of Tadpole* $T(5, 3)$ *is shown in Figure 5.*

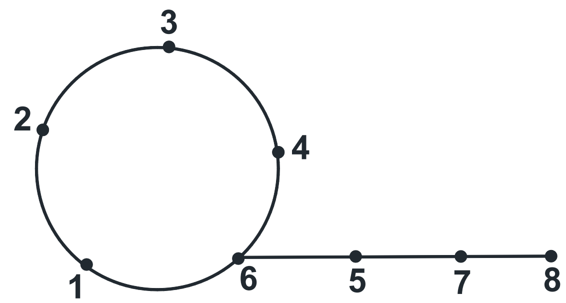

Fig. 5

Theorem II.4. *Middle graph of cycle* (C_n) *is strongly*-graph for all* $n \in \mathbb{N}$.

Proof. Let $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$ be the vertices of the graph $M(C_n)$, where $\{v_1, v_2, \ldots, v_n\}$ be the vertices corresponding to the cycle C_n and $\{v'_1, v'_2, \ldots, v'_n\}$ be the vertices corresponding to the edges of C_n.

We define vertex labeling function $f : V(M(C_n)) \to \{1, 2, 3, \ldots, 2n\}$ as follows.

$$f(v_i) = \begin{cases} 4i - 3; 1 \leq i \leq \lceil \frac{n}{2} \rceil. \\ 4(n - i + 1); \lceil \frac{n}{2} \rceil + 1 \leq i \leq n. \end{cases}$$

$$f(v'_i) = \begin{cases} 4i - 1; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor. \\ 4(n - i) + 2; \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n. \end{cases}$$

Here vertex labels in one half of the $M(C_n)$ are consecutive even numbers in increasing order, whereas the vertex labels in other half of the $M(C_n)$ are consecutive odd numbers in increasing order.

When n is even, for any even edge label produced in graph, $\{(4k)+(4k+2)+(4k)\cdot(4k+2)\} \neq \{(4k+2)+(4(k+1))+(4k+2)\cdot(4(k+1))\} \neq \{(4(k+1))+(4k)+(4(k+1))\cdot(4k)\} \neq \{(2)+(4)+(2)\cdot(4)\}$ and for any odd edge label produced in the graph, $\{(4k-3)+(4k-1)+(4k-3)\cdot(4k-1)\} \neq \{(4k-1)+(4k+1)+(4k-1)\cdot(4k+1)\} \neq \{(4k+1)+(4k-3)+(4k+1)\cdot(4k-3)\} \neq \{(2n-3)+(2n-1)+(2n-3)\cdot(2n-1)\} \neq \{(2n-1)+(2n)+(2n-1)\cdot(2n)\} \neq \{(2n-3)+(2n)+(2n-3)\cdot(2n)\} \neq \{(1)+(2)+(1)\cdot(2)\} \neq \{(1)+(4)+(1)\cdot(4)\}$. where $1 \leq k \leq \frac{n}{2}-1$
.

When n is odd, for any even edge label produced in the graph $\{(4(k)+(4k+2)+(4k)\cdot(4k+2)\} \neq \{(4k+2)+(4(k+1))+(4k+2)\cdot(4(k+1))\} \neq (4(k+1))+(4k)+(4(k+1))\cdot(4k)\} \neq \{(2n-2)+(2n)+(2n-2)\cdot(2n)\} \neq \{(2)+(4)+(2)\cdot(4)\}$, where $1 \leq k \leq \lfloor\frac{n}{2}\rfloor - 1$ and for any odd edge label produced in graph $\{(4k-3)+(4k-1)+(4k-3)\cdot(4k-1)\} \neq \{(4k-1)+(4k+1)+(4k-1)\cdot(4k+1)\} \neq \{(4k+1)+(4k-3)+(4k+1)\cdot(4k-3)\} \neq \{(2n-2)+(2n-1)+(2n-2)\cdot(2n-1)\} \neq \{(2n-1)+(2n)+(2n-1)\cdot(2n)\} \neq \{(1)+(2)+(1)\cdot(2)\} \neq \{(1)+(4)+(1)\cdot(4)\}$, where $1 \leq k \leq \lfloor\frac{n}{2}\rfloor$
.

Hence above defined labeling pattern satisfies the conditions of strongly*-labeling. i.e. $M(C_n)$ is strongly*-graph. □

Example 6. *Strongly*-labeling of* $M(C_4)$ *is shown in Figure 6 .*

Theorem II.5. *Middle graph of path* P_n *is strongly*-graph for all* $n \in \mathbb{N}$.

Proof. Let $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_{n-1}\}$ be the vertices of the $M(P_n)$, where $\{v_1, v_2, \ldots, v_n\}$ be the vertices corresponding to path P_n and $\{v'_1, v'_2, \ldots, v'_{n-1}\}$ be the vertices corresponding to the edges $\{e_1, e_2, \ldots, e_{n-1}\}$ of P_n.

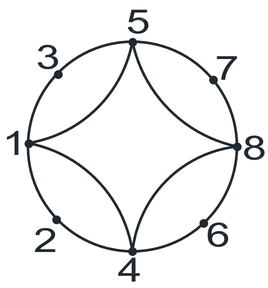

Fig. 6

We define vertex labeling $f : V(M(P_n)) \to \{1, 2, 3, \ldots, 2n-1\}$ as follows.
$f(v_i) = 2i-1,\ 1 \leq i \leq n.$
$f(v'_i) = 2i,\ 1 \leq i \leq n-1.$
The labels for the vertices v_i are consecutive odd numbers, whereas the labels for the vertices v'_i are consecutive even numbers. Therefore the labels produced for the edges $v_iv'_i$ are odd and in increasing order. Further the edges $v'_iv'_{i+1}$ and $(1 \leq i \leq n-2)$ are labeled with even label in increasing order. Hence all edges will have different label. So, labeling pattern defined above satisfies the conditions of strongly*-labeling. i.e. $M(P_n)$ is strongly*-graph. □

Example 7. *Strongly*-labeling of* $M(P_7)$ *is shown in Figure 7 .*

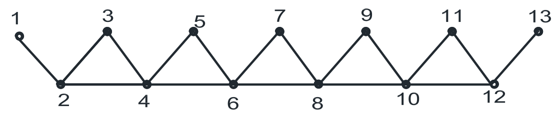

Fig. 7

Theorem II.6. *The total graph of P_n is strongly*-graph for all $n \in \mathbb{N}$.*

Proof. Let $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_{n-1}\}$ be the vertices of $T(P_n)$, where $\{v_1, v_2, \ldots, v_n\}$ be the vertices of path P_n and $\{v'_1, v'_2, \ldots, v'_{n-1}\}$ be the vertices corresponding to the edges $\{e_1, e_2, \ldots, e_{n-1}\}$ of P_n.
We define vertex labeling function $f : V(T(P_n)) \to \{1, 2, 3, \ldots, 2n-1\}$ as follows.
$f(v_i) = 2i - 1$, $1 \le i \le n$.
$f(v'_i) = 2i$, $1 \le i \le n-1$.
The labels for the vertices v_i are consecutive odd numbers, whereas the labels for the vertices v'_i are consecutive even numbers. Therefore the labels produced for the edges $v_i v'_i$ and $v_i v_{i+1}$ are odd and in increasing order. For the label of any of the remarking edges with common end vertices, we have $\{(2i-1)+(2i+1)+(2i-1)\cdot(2i+1)\} \neq \{(2i)+(2i+1)+(2i)\cdot(2i+1)\} \neq \{(2i)+(2i-1)+(2i)\cdot(2i-1)\}$, $1 \le i \le n-1$. Further the edges incident with v'_i and v'_{i+1} $(1 \le i \le n-2)$ are labeled with even labels in increasing order. Hence all edges will have different labels. So, labeling pattern defined above satisfies the conditions of strongly*-labeling. i.e. $M(P_n)$ is strongly*-graph. □

Example 8. *Strongly*-labeling of $T(P_5)$ is shown in Figure 8 as an illustration for Theorem 6.*

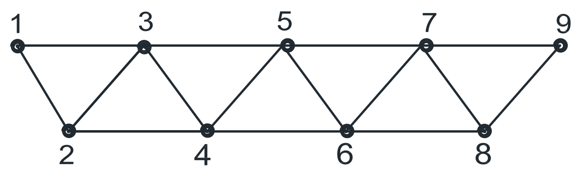

Fig. 8

Theorem II.7. *The split graph of P_n is strongly*-graph for all $n \in \mathbb{N}$.*

Proof. Let $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the vertices of the $spl(P_n)$, where $\{v_1, v_2, \ldots, v_n\}$ are the vertices of the path P_n and $\{u_1, u_2, \ldots, u_n\}$ are newly added vertices corresponding to the vertices of P_n to obtain $spl(P_n)$.
We define vertex labeling function $f : V(spl(P_n)) \to \{1, 2, 3, \ldots, 2n\}$ as follows.
$f(v_i) = 2i - 1$, $1 \le i \le n$.
$f(u_i) = 2i$, $1 \le i \le n$.
Here for the label of any two edges with common vertex, we have $\{(2i-1)+(2i+1)+(2i-1)\cdot(2i+1)\} \neq \{(2i-1)+(2i+2)+(2i-1)\cdot(2i+2)\} \neq \{2i+(2i+1)+2i\cdot(2i+1)\}$. Therefor above defined labeling pattern satisfies the conditions of strongly*-graph. i.e. $spl(P_n)$ is strongly*-graph. □

Example 9. *Strongly*-labeling of $spl(P_7)$ is shown in Figure 9 as an illustration for Theorem 7.*

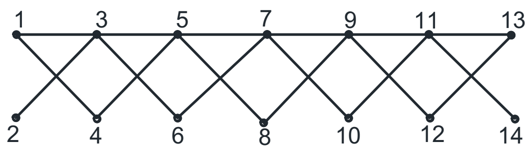

Fig. 9

Theorem II.8. *The shadow graph $D_2(P_n)$ is strongly*-graph for all $n \in \mathbb{N}$.*

Proof. Let $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$ be the verties of the $D_2(P_n)$, where $\{v_1, v_2, \ldots, v_n\}$ are the vertices of path P_n and $\{v'_1, v'_2, \ldots, v'_n\}$ are the vertices added corresponding to the vertices $\{v_1, v_2, \ldots, v_n\}$ in order to obtain $D_2(P_n)$.
We define vertex labeling $f : V(D_2(P_n)) \to \{1, 2, 3, \ldots, 2n\}$ as follows.
$f(v_i) = 2i - 1$, $1 \le i \le n$.
$f(v'_i) = 2i$, $1 \le i \le n$.

The labels for the vertices v_i are consecutive odd numbers, whereas the labels for the vertices v_i' are consecutive even numbers. Therefore the labels produced for the edges $v_i'v_{i+1}'$ are even labels in increasing order the labels for edges v_iv_{i+1}, v_iv_{i+1}' and $v_i'v_{i+1}$ are odd labels in increasing order. Also for any two edge labels produce in the graph, $\{(2i-1)+(2i+1)+(2i-1)\cdot(2i+1)\} \neq \{(2i-1)+(2i+2)+(2i-1)\cdot(2i+2)\} \neq \{2i+(2i+1)+2i\cdot(2i+1)\}$ for any i. So, the labeling pattern defined above satisfies the conditions of strongly*-graph. i.e. $D_2(P_n)$ is strongly*-graph. □

Example 10. *Strongly*-labeling of $D_2(P_6)$ is shown in Figure 10 as an illustration for Theorem 8.*

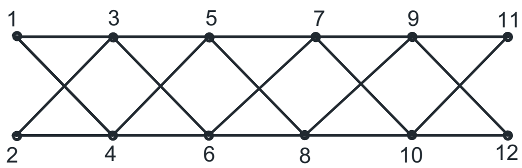

Fig. 10

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