$(1,2)^{\star}$-ψ-LOCALLY CLOSED SETS

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ABSTRACT. In this paper, we introduce three forms of $(1,2)^{\star}$-locally closed sets called $(1,2)^{\star}$-ψ-locally closed sets, $(1,2)^{\star}$-ψ-lc*-sets and $(1,2)^{\star}$-ψ-lc**-sets. Properties of these new concepts are studied as well as their relations to the other classes of $(1,2)^{\star}$-locally closed sets will be investigated.

1. Introduction

The first step of locally closedness was done by Bourbaki [2]. He defined a set A to be locally closed if it is the intersection of an open set and a closed set. In literature many general topologists introduced the studies of locally closed sets. Extensive research on locally closedness and generalizing locally closedness were done in recent years. Stone [15] used the term FG for a locally closed set. Ganster and Reilly used

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locally closed sets in [4] to define LC-continuity and LC-irresoluteness. Balachandran et al [1] introduced the concept of generalized locally closed sets. Veera Kumar [16] (Sheik John [14]) introduced $\hat{g}$-locally closed sets (= ω-locally closed sets) respectively.

In this paper, we introduce three forms of $(1,2)^{\star}$-locally closed sets called $(1,2)^{\star}$-ψ-locally closed sets, $(1,2)^{\star}$-ψ-lc*-sets and $(1,2)^{\star}$-ψ-lc**-sets. Properties of these new concepts are studied as well as their relations to the other classes of $(1,2)^{\star}$-locally closed sets will be investigated.

2. Preliminaries

Definition 2.1. *Let S be a subset of X. Then S is said to be $\tau_{1,2}$-open* [8] *if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$.*

The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed.

Notice that $\tau_{1,2}$-open sets need not necessarily form a topology.

Definition 2.2. [8] *Let S be a subset of a bitopological space X. Then*

(1) *the $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-cl(S), is defined as $\cap \{F : S \subseteq F$ and F is $\tau_{1,2}$-closed$\}$.*

(2) *the $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-int(S), is defined as $\cup \{F : F \subseteq S$ and F is $\tau_{1,2}$-open$\}$.*

Definition 2.3. *A subset A of a bitopological space X is called*

(1) *$(1,2)^{\star}$-semi-open set* [7] *if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int(A));*

(2) *$(1,2)^{\star}$-α-open set* [6] *if $A \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int(A)));*

(3) *$(1,2)^{\star}$-β-open set* [10] *if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl(A))).*

Definition 2.4. *A subset A of a bitopological space (X, τ_1, τ_2) is called*

(1) *$(1,2)^{\star}$-g-closed set* [11] *if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X. The complement of $(1,2)^{\star}$-g-closed set is called $(1,2)^{\star}$-g-open set;*

(2) *$(1,2)^{\star}$-sg-closed set* [7] *if $(1,2)^{\star}$-scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-semi-open in X. The complement of $(1,2)^{\star}$-sg-closed set is called $(1,2)^{\star}$-sg-open set;*

(3) *$(1,2)^{\star}$-$\hat{g}$-closed set* [3] *or $(1,2)^{\star}$-ω-closed set* [5] *if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-semi-open in X. The complement of $(1,2)^{\star}$-$\hat{g}$-closed (resp. $(1,2)^{\star}$-ω-closed) set is called $(1,2)^{\star}$-$\hat{g}$-open (resp. $(1,2)^{\star}$-ω-open) set;*

(4) *$(1,2)^{\star}$-ψ-closed set* [12] *if $(1,2)^{\star}$scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-sg-open in X. The complement of $(1,2)^{\star}$-ψ-closed set is called $(1,2)^{\star}$-ψ-open set.*

Definition 2.5. *A subset A of a bitopological space X is called*

(1) *regular $(1,2)^{\star}$-open* [9] *if $A = \tau_{1,2}$-int($\tau_{1,2}$-cl(A)).*

(2) *$(1,2)^{\star}$-regular generalized closed (briefly, $(1,2)^{\star}$-rg-closed) set* [5] *if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is regular $(1,2)^{\star}$-open in X.*

The complement of $(1,2)^{\star}$-rg-closed set is called $(1,2)^{\star}$-rg-open set.

Remark 2.6. *The collection of all $(1,2)^{\star}$-rg-closed sets in X is denoted by $(1,2)^{\star}$-RGC(X).*

The collection of all $(1,2)^{\star}$-rg-open sets in X is denoted by $(1,2)^{\star}$-RGO(X).

We denote the power set of X by P(X).

Corollary 2.7. *If A is a $(1,2)^{\star}$-ψ-closed set and F is a $\tau_{1,2}$-closed set, then $A \cap F$ is a $(1,2)^{\star}$-ψ-closed set.*

Proposition 2.8. [13] *Every $\tau_{1,2}$-closed set is $(1,2)^{\star}$-ψ-closed.*

Proposition 2.9. [13] *Every $(1,2)^{\star}$-ψ-closed set is $(1,2)^{\star}$-$\hat{g}$-closed.*

Proposition 2.10. [13] *Every* $(1,2)^{\star}$-ψ-*closed set is* $(1,2)^{\star}$-*g-closed.*

Proposition 2.11. [13] *Every* $(1,2)^{\star}$-ψ-*closed set is* $(1,2)^{\star}$-*sg-closed.*

3. $(1,2)^{\star}$-ψ-locally closed sets

We introduce the following definition.

Definition 3.1. *A subset S of a bitopological space X is called*

(1) $(1,2)^{\star}$-*locally closed (briefly,* $(1,2)^{\star}$-*lc) if* $S = U \cap F$, *where U is* $\tau_{1,2}$-*open and F is* $\tau_{1,2}$-*closed in X.*

(2) $(1,2)^{\star}$-*generalized locally closed (briefly,* $(1,2)^{\star}$-*glc) if* $S = U \cap F$, *where U is* $(1,2)^{\star}$-*g-open and F is* $(1,2)^{\star}$-*g-closed in X.*

(3) $(1,2)^{\star}$-*semi-generalized locally closed (briefly,* $(1,2)^{\star}$-*sglc) if* $S = U \cap F$, *where U is* $(1,2)^{\star}$-*sg-open and F is* $(1,2)^{\star}$-*sg-closed in X.*

(4) $(1,2)^{\star}$-*regular-generalized locally closed (briefly,* $(1,2)^{\star}$-*rg-lc) if* $S = U \cap F$, *where U is* $(1,2)^{\star}$-*rg-open and F is* $(1,2)^{\star}$-*rg-closed in X.*

(5) *generalized locally* $(1,2)^{\star}$-*semi-closed (briefly,* $(1,2)^{\star}$-*glsc) if* $S = U \cap F$, *where U is* $(1,2)^{\star}$-*g-open and F is* $(1,2)^{\star}$-*semi-closed in X.*

(6) $(1,2)^{\star}$-*locally semi-closed (briefly,* $(1,2)^{\star}$-*lsc) if* $S = U \cap F$, *where U is* $\tau_{1,2}$-*open and F is* $(1,2)^{\star}$-*semi-closed in X.*

(7) $(1,2)^{\star}$-α-*locally closed (briefly,* $(1,2)^{\star}$-α-*lc) if* $S = U \cap F$, *where U is* $(1,2)^{\star}$-α-*open and F is* $(1,2)^{\star}$-α-*closed in X.*

(8) $(1,2)^{\star}$-ω-*locally closed (briefly,* $(1,2)^{\star}$-ω-*lc) if* $S = U \cap F$, *where U is* $(1,2)^{\star}$-ω-*open and F is* $(1,2)^{\star}$-ω-*closed in X.*

(9) $(1,2)^{\star}$-*sglc* if* $S = U \cap F$, *where U is* $(1,2)^{\star}$-*sg-open and F is* $\tau_{1,2}$-*closed in X.*

The class of all $(1,2)^{\star}$-locally closed (resp. $(1,2)^{\star}$-generalized locally closed, $(1,2)^{\star}$-generalized locally semi-closed, $(1,2)^{\star}$-locally semi-closed, $(1,2)^{\star}$-ω-locally closed) sets in X is denoted by $(1,2)^{\star}$-LC(X) (resp. $(1,2)^{\star}$-GLC(X), $(1,2)^{\star}$-GLSC(X), $(1,2)^{\star}$-LSC(X), $(1,2)^{\star}$-ω-LC(X)).

Definition 3.2. *A subset of a bitopological space X is called $(1,2)^{\star}$-ψ-locally closed (briefly, $(1,2)^{\star}$-ψ-lc) if $A = S \cap G$, where S is $(1,2)^{\star}$-ψ-open and G is $(1,2)^{\star}$-ψ-closed in X.*

The class of all $(1,2)^{\star}$-ψ-locally closed sets in X is denoted by $(1,2)^{\star}$- $\psi LC(X)$.

Proposition 3.3. *Every $(1,2)^{\star}$-ψ-closed (resp. $(1,2)^{\star}$-ψ-open) set is $(1,2)^{\star}$-ψ-lc set but not conversely.*

Proof. It follows from Definition 3.2.

Example 3.4. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{b\}\}$ and $\tau_2 = \{\emptyset,\ X\}$. Then the sets in $\{\emptyset,\ X,\ \{b\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset,\ X,\ \{a,\ c\}\}$ are called $\tau_{1,2}$-closed. Then the set $\{b\}$ is $(1,2)^{\star}$-ψ-lc set but it is not $(1,2)^{\star}$-ψ-closed and the set $\{a,\ c\}$ is $(1,2)^{\star}$-ψ-lc set but it is not $(1,2)^{\star}$-ψ-open in X.*

Proposition 3.5. *Every $(1,2)^{\star}$-lc set is $(1,2)^{\star}$-ψ-lc set but not conversely.*

Proof. It follows from Proposition 2.8.

Example 3.6. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{b,\ c\}\}$ and $\tau_2 = \{\emptyset,\ X\}$. Then the sets in $\{\emptyset,\ X,\ \{b,\ c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset,\ X,\ \{a\}\}$ are called $\tau_{1,2}$-closed. Then the set $\{b\}$ is $(1,2)^{\star}$-ψ-lc set but it is not$(1,2)^{\star}$-lc set in X.*

Proposition 3.7. *Every $(1,2)^{\star}$-ψ-lc set is a (i) $(1,2)^{\star}$-ω-lc set, (ii) $(1,2)^{\star}$-glc set and (iii) $(1,2)^{\star}$-sglc set. However the separate converses are not true.*

Proof. It follows from Propositions 2.9, 2.10 and 2.11.

Example 3.8. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{a\}\}$ *and* $\tau_2 = \{\emptyset,\ X,\ \{b,\ c\}\}$. *Then the sets in* $\{\emptyset,\ X,\ \{a\},\ \{b,\ c\}\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\emptyset,\ X,\ \{a\},\ \{b,\ c\}\}$ *are called* $\tau_{1,2}$*-closed. Then the set* $\{b\}$ *is* $(1,2)^{\star}$*-g-lc set but it is not* $(1,2)^{\star}$*-*ψ*-lc set in X. Moreover, the set* $\{c\}$ *is* $(1,2)^{\star}$*-sg-lc set but it is not* $(1,2)^{\star}$*-*ψ*-lc set in X.*

Example 3.9. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{b\}\}$ *and* $\tau_2 = \{\emptyset,\ X,\ \{a,\ c\}\}$. *Then the sets in* $\{\emptyset,\ X,\ \{b\},\ \{a,\ c\}\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\emptyset,\ X,\ \{b\},\ \{a,\ c\}\}$ *are called* $\tau_{1,2}$*-closed. Then the set* $\{a\}$ *is* $(1,2)^{\star}$*-*ω*-lc set but it is not* $(1,2)^{\star}$*-*ω*-lc set in X.*

Proposition 3.10. *Every* $(1,2)^{\star}$*-*α*-lc set is* $(1,2)^{\star}$*-*ψ*-lc set but not conversely.*

Example 3.11. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{a,\ c\}\}$ *and* $\tau_2 = \{\emptyset,\ X\}$. *Then the sets in* $\{\emptyset,\ X,\ \{a,\ c\}\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\emptyset,\ X,\ \{b\}\}$ *are called* $\tau_{1,2}$*-closed. Then the set* $\{a\}$ *is* $(1,2)^{\star}$*-*ψ*-lc set but it is not* $(1,2)^{\star}$*-*α*-lc set in X.*

Proposition 3.12. *Every* $(1,2)^{\star}$*-lsc set is* $(1,2)^{\star}$*-*ψ*-lc set but not conversely.*

Example 3.13. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{a,\ c\}\}$ *and* $\tau_2 = \{\emptyset,\ X\}$. *Then the sets in* $\{\emptyset,\ X,\ \{a,\ c\}\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\emptyset,\ X,\ \{b\}\}$ *are called* $\tau_{1,2}$*-closed. Then the set* $\{a\}$ *is* $(1,2)^{\star}$*-*ψ*-lc set but it is not* $(1,2)^{\star}$*-lsc set in X.*

Proposition 3.14. *Every* $(1,2)^{\star}$*-glsc set is* $(1,2)^{\star}$*-*ψ*-lc set but not conversely.*

Example 3.15. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{a,\ c\}\}$ *and* $\tau_2 = \{\emptyset,\ X\}$. *Then the sets in* $\{\emptyset,\ X,\ \{a,\ c\}\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\emptyset,\ X,\ \{b\}\}$ *are called* $\tau_{1,2}$*-closed. Then the set* $\{a,\ b\}$ *is* $(1,2)^{\star}$*-*ψ*-lc set but it is not* $(1,2)^{\star}$*-glsc set in X.*

Proposition 3.16. *Every* $(1,2)^{\star}$*-sglc* set is* $(1,2)^{\star}$*-*ψ*-lc set but not conversely.*

Example 3.17. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{a,\ c\}\}$ and $\tau_2 = \{\emptyset,\ X\}$. Then the sets in $\{\emptyset,\ X,\ \{a,\ c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset,\ X,\ \{b\}\}$ are called $\tau_{1,2}$-closed. Then the set $\{a,\ b\}$ is $(1,2)^{\star}$-ψ-lc set but it is not $(1,2)^{\star}$-sglc* set in X.*

Theorem 3.18. *For a $T(1,2)^{\star}$-ψ-space X, the following properties hold:*

(1) *$(1,2)^{\star}$-ψLC(X) $=$ $(1,2)^{\star}$-LC(X).*

(2) *$(1,2)^{\star}$-ψLC(X) $\subseteq$ $(1,2)^{\star}$-GLC(X).*

(3) *$(1,2)^{\star}$-ψLC(X) $\subseteq$ $(1,2)^{\star}$-GLSC(X).*

(4) *$(1,2)^{\star}$-ψLC(X) $\subseteq$ $(1,2)^{\star}$-ω-LC(X).*

Proof. (1) Since every $(1,2)^{\star}$-ψ-open set is $\tau_{1,2}$-open and every $(1,2)^{\star}$-ψ-closed set is $\tau_{1,2}$-closed in X, $(1,2)^{\star}$-ψLC(X) $\subseteq$ $(1,2)^{\star}$-LC(X) and hence $(1,2)^{\star}$-ψLC(X) $=$ $(1,2)^{\star}$-LC(X).

(2), (3) and (4) follows from (1), since for any space (X, τ), $(1,2)^{\star}$-LC(X) $\subseteq$ $(1,2)^{\star}$-GLC(X), $(1,2)^{\star}$-LC(X) $\subseteq$ $(1,2)^{\star}$-GLSC(X) and $(1,2)^{\star}$-LC(X) $\subseteq$ $(1,2)^{\star}$-ω-LC(X).

Corollary 3.19. *If $(1,2)^{\star}$-GO(X) $=$ $(1,2)^{\star}$-O(X) where $(1,2)^{\star}$-O(X) is the collection of all $\tau_{1,2}$-open subsets of X, then $(1,2)^{\star}$-ψLC(X) $\subseteq$ $(1,2)^{\star}$-GLSC(X) $\subseteq$ $(1,2)^{\star}$-LSC(X).*

Proof. $(1,2)^{\star}$-GO(X) $=$ $(1,2)^{\star}$-O(X) implies that X is a T$(1,2)^{\star}$-ψ-space and hence by Theorem 3.18, $(1,2)^{\star}$-ψLC(X) $\subseteq$ $(1,2)^{\star}$-GLSC(X). Let A $\in$ $(1,2)^{\star}$-GLSC(X). Then A $=$ U $\cap$ F, where U is $(1,2)^{\star}$-g-open and F is $(1,2)^{\star}$-semi-closed. By hypothesis, U is $\tau_{1,2}$-open and hence A is a $(1,2)^{\star}$-lsc set and so A $\in$ $(1,2)^{\star}$-LSC(X).

Definition 3.20. *A subset A of a bitopological space X is called*

(1) *$(1,2)^{\star}$-ψ-lc* set if A$=$ S $\cap$ G, where S is $(1,2)^{\star}$-ψ-open in X and G is $\tau_{1,2}$-closed in X.*

(2) $(1,2)^{\star}$-ψ-*lc*** *set if* $A= S \cap G$, *where* S *is* $\tau_{1,2}$-*open in* X *and* G *is* $(1,2)^{\star}$-ψ-*closed in* X.

The class of all $(1,2)^{\star}$-ψ-*lc** *(resp.* $(1,2)^{\star}$-ψ-*lc**) *sets in a bitopological space* X *is denoted by* $(1,2)^{\star}$-$\psi LC^*(X)$ *(resp.* $(1,2)^{\star}$-$\psi LC^{**}(X)$*).*

Proposition 3.21. *Every* $(1,2)^{\star}$-*lc set is* $(1,2)^{\star}$-ψ-*lc** *set but not conversely.*

Proof. It follows from Definitions 3.1 (1) and 3.20 (1).

Example 3.22. *The set* $\{b\}$ *in Example 3.6 is* $(1,2)^{\star}$-ψ-*lc** *set but it is not a* $(1,2)^{\star}$-*lc set in* X.

Proposition 3.23. *Every* $(1,2)^{\star}$-*lc set is* $(1,2)^{\star}$-ψ-*lc*** *set but not conversely.*

Proof. It follows from Definitions 3.1 (1) and 3.20 (2).

Example 3.24. *The set* $\{a,\ c\}$ *in Example 3.6 is* $(1,2)^{\star}$-ψ-*lc*** *set but it is not a* $(1,2)^{\star}$-*lc set in* X.

Proposition 3.25. *Every* $(1,2)^{\star}$-ψ-*lc** *set is* $(1,2)^{\star}$-ψ-*lc set but not conversely.*

Proof. It follows from Definitions 3.2 and 3.20 (1).

Example 3.26. *The set* $\{a,\ b\}$ *in Example 3.6 is* $(1,2)^{\star}$-ψ-*lc set but it is not a* $(1,2)^{\star}$-ψ-*lc** *set in* X.

Proposition 3.27. *Every* $(1,2)^{\star}$-ψ-*lc*** *set is* $(1,2)^{\star}$-ψ-*lc set but not conversely.*

Proof. It follows from Definitions 3.2 and 3.20 (2).

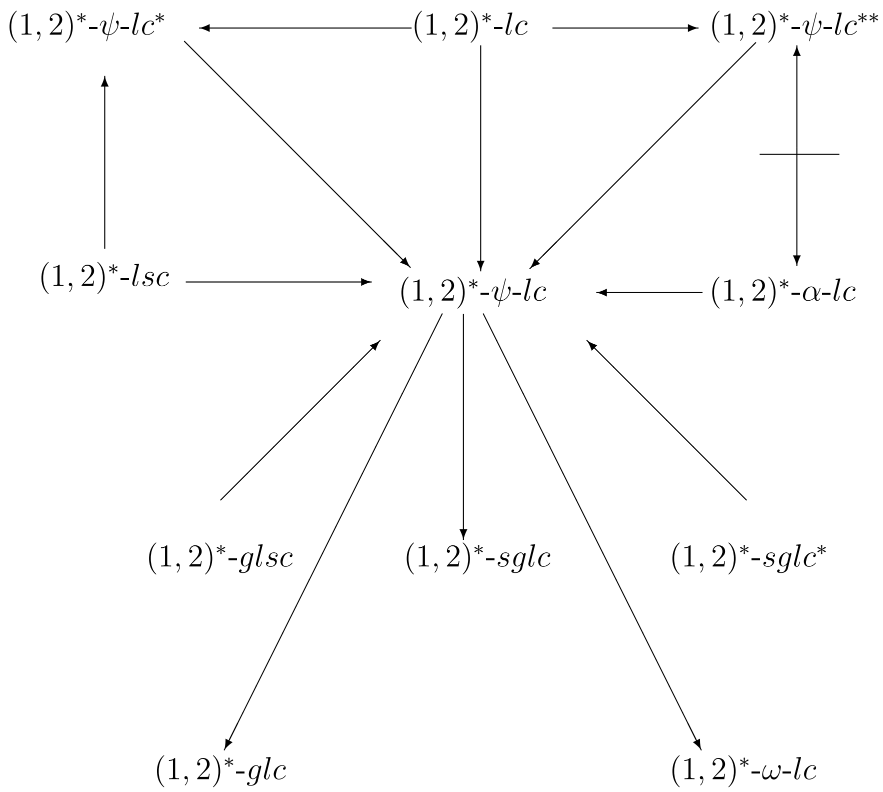
Remark 3.28. *The concepts of* $(1,2)^{\star}$-ψ-*lc** *sets and* $(1,2)^{\star}$-*lsc sets are independent of each other.*

Example 3.29. *The set* $\{c\}$ *in Example 3.6 is* $(1,2)^{\star}$-ψ-*lc** *set but it is not a* $(1,2)^{\star}$-*lsc set in* X *and the set* $\{a\}$ *in Example 3.4 is* $(1,2)^{\star}$-*lsc set but it is not a* $(1,2)^{\star}$-ψ-*lc** *set in* X.

Remark 3.30. *The concepts of* $(1,2)^{\star}$-ψ-*lc*** *sets and* $(1,2)^{\star}$-α-*lc sets are independent of each other.*

Example 3.31. *The set* $\{a,\ b\}$ *in Example 3.6 is* $(1,2)^{\star}$-ψ-*lc*** *set but it is not a* $(1,2)^{\star}$-α-*lc set in* X *and the set* $\{a,\ b\}$ *in Example 3.4 is* $(1,2)^{\star}$-α-*lc set but it is not a* $(1,2)^{\star}$-ψ-*lc** *set in* X.

Remark 3.32. *From the above discussions we have the following implications where* $A \rightarrow B$ *(resp.* $A\ B$*) represents* A *implies* B *but not conversely (resp.* A *and* B *are independent of each other).*



Proposition 3.33. *If* $(1,2)^\star$*-GO(X)* $=$ $(1,2)^\star$*-O(X), then* $(1,2)^\star$*-*ψ*LC(X)* $=$ $(1,2)^\star$*-*ψ*LC*(X)* $=$ $(1,2)^\star$*-*ψ*LC**(X).*

Proof. Since $(1,2)^\star$-ψO(X) $\subseteq$ $(1,2)^\star$-GO(X) $=$ $(1,2)^\star$-O(X), therefore by hypothesis, X is a T$(1,2)^\star$-ψ-space and hence $(1,2)^\star$-ψLC(X) $=$ $(1,2)^\star$-ψLC*(X) $=$ $(1,2)^\star$-ψLC**(X).

Remark 3.34. *The converse of Proposition 3.33 need not be true.*

For the bitopological space X in Example 3.4, $(1,2)^\star$*-*ψ*LC(X)* $=$ $(1,2)^\star$*-*ψ*LC*(X)* $=$ $(1,2)^\star$*-*ψ*LC**(X). However* $(1,2)^\star$*-GO(X)* $= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$ $\neq$ $(1,2)^\star$*-O(X).*

Proposition 3.35. *Let X be a bitopological space. If* $(1,2)^{\star}$*-GO(X)* $\subseteq$ $(1,2)^{\star}$*-LC(X), then* $(1,2)^{\star}$*-*ψ*LC(X)* $=$ $(1,2)^{\star}$*-*ψ*LC**(X).*

Proof. Let A $\in$ $(1,2)^{\star}$-ψLC(X). Then A $=$ S $\cap$ G where S is $(1,2)^{\star}$-ψ-open and G is $(1,2)^{\star}$-ψ-closed. Since $(1,2)^{\star}$-GO(X)$\subseteq$ $(1,2)^{\star}$-GO(X) and by hypothesis $(1,2)^{\star}$-GO(X) $\subseteq$ $(1,2)^{\star}$-LC(X), S is $(1,2)^{\star}$-locally closed. Then S $=$ P $\cap$ Q, where P is $\tau_{1,2}$-open and Q is $\tau_{1,2}$-closed. Therefore, A $=$ P $\cap$ (Q $\cap$ G). By Corollary 2.7, Q $\cap$ G is $(1,2)^{\star}$-ψ-closed and hence A $\in$ $(1,2)^{\star}$-ψLC**(X). That is $(1,2)^{\star}$-ψLC(X) $\subseteq$ $(1,2)^{\star}$-ψLC**(X). For any bitopological space, $(1,2)^{\star}$-ψLC**(X) $\subseteq$ $(1,2)^{\star}$-ψLC(X) and so $(1,2)^{\star}$-ψLC(X) $=$ $(1,2)^{\star}$-ψLC**(X).

Remark 3.36. *The converse of Proposition 3.35 need not be true in general.*

For the bitopological space X in Example 3.4, then $(1,2)^{\star}$*-*ψ*LC(X)* $=$ $(1,2)^{\star}$*-*ψ*LC**(X)* $= \{\emptyset, \{b\}, \{a, c\}, X\}$*. But* $(1,2)^{\star}$*-GO(X)* $= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\} \nsubseteq$ $(1,2)^{\star}$*-*ψ*LC(X)* $= \{\emptyset, \{b\}, \{a, c\}, X\}$.

Corollary 3.37. *Let X be a bitopological space. If* $(1,2)^{\star}$*-*ω*O(X)* $\subseteq$ $(1,2)^{\star}$*-LC(X), then* $(1,2)^{\star}$*-*ψ*LC(X)* $=$ $(1,2)^{\star}$*-*ψ*LC**(X).*

Proof. It follows from the fact that $(1,2)^{\star}$-ωO(X) $\subseteq$ $(1,2)^{\star}$-GO(X) and Proposition 6.3.35.

Remark 3.38. *The converse of Corollary 3.37 need not be true in general.*

For the bitopological space X in Example 3.6, then $(1,2)^{\star}$*-*ψ*LC(X)* $=$ $(1,2)^{\star}$*-*ψ*LC**(X)* $=$ *P(X). But* $(1,2)^{\star}$*-*ω*O(X)* $=$ *P(X)*$\nsubseteq$ $(1,2)^{\star}$*-LC(X)* $= \{\emptyset, \{a\}, \{b, c\}, X\}$.

The following theorems are exploring the characterizations of $(1,2)^{\star}$*-*ψ*-lc sets,* $(1,2)^{\star}$*-*ψ*-lc* sets and* $(1,2)^{\star}$*-*ψ*-lc** sets.*

Theorem 3.39. *For a subset A of X the following statements are equivalent:*

(1) $A \in (1,2)^{\star}$-$\psi LC(X)$,

(2) $A = S \cap (1,2)^{\star}$-ψ-$cl(A)$ *for some* $(1,2)^{\star}$-ψ-*open set* S,

(3) $(1,2)^{\star}$-ψ-$cl(A) - A$ *is* $(1,2)^{\star}$-ψ-*closed*,

(4) $A \cup ((1,2)^{\star}$-ψ-$cl(A))^c$ *is* $(1,2)^{\star}$-ψ-*open*,

(5) $A \subseteq (1,2)^{\star}$-$\psi$-$int(A \cup ((1,2)^{\star}$-$\psi$-$cl(A))^c)$.

Proof. (1) $\Rightarrow$ (2). Let $A \in (1,2)^{\star}$-ψLC(X). Then $A = S \cap G$ where S is $(1,2)^{\star}$-ψ-open and G is $(1,2)^{\star}$-ψ-closed. Since $A \subseteq G$, $(1,2)^{\star}$-ψ-cl(A) $\subseteq$ G and so $S \cap (1,2)^{\star}$-ψ-cl(A) $\subseteq$ A. Also $A \subseteq S$ and $A \subseteq (1,2)^{\star}$-$\psi$-cl(A) implies $A \subseteq S \cap (1,2)^{\star}$-$\psi$-cl(A) and therefore $A = S \cap (1,2)^{\star}$-ψ-cl(A).

(2) $\Rightarrow$ (3). $A = S \cap (1,2)^{\star}$-ψ-cl(A) implies $(1,2)^{\star}$-ψ-cl(A) $-$ A $= (1,2)^{\star}$-ψ-cl(A) $\cap S^c$ which is $(1,2)^{\star}$-ψ-closed since S^c is $(1,2)^{\star}$-ψ-closed and $(1,2)^{\star}$-ψ-cl(A) is $(1,2)^{\star}$-ψ-closed.

(3) $\Rightarrow$ (4). $A \cup ((1,2)^{\star}$-ψ-cl(A))^c = $((1,2)^{\star}$-ψ-cl(A) $-$ A$)^c$ and by assumption, $((1,2)^{\star}$-ψ-cl(A) $-$ A$)^c$ is $(1,2)^{\star}$-ψ-open and so is $A \cup ((1,2)^{\star}$-ψ-cl(A))^c.

(4) $\Rightarrow$ (5). By assumption, $A \cup ((1,2)^{\star}$-ψ-cl(A))^c = $(1,2)^{\star}$-ψ-int($A \cup ((1,2)^{\star}$-ψ-cl(A))^c) and hence $A \subseteq (1,2)^{\star}$-$\psi$-int($A \cup ((1,2)^{\star}$-$\psi$-cl(A))^c).

(5) $\Rightarrow$ (1). By assumption and since $A \subseteq (1,2)^{\star}$-$\psi$-cl(A), $A = (1,2)^{\star}$-ψ-int($A \cup ((1,2)^{\star}$-ψ-cl(A))c) $\cap (1,2)^{\star}$-ψ-cl(A). Therefore, $A \in (1,2)^{\star}$-ψLC(X).

Theorem 3.40. *For a subset A of X, the following statements are equivalent:*

(1) $A \in (1,2)^{\star}$-$\psi LC^*(X)$,

(2) $A = S \cap \tau_{1,2}$-$cl(A)$ *for some* $(1,2)^{\star}$-ψ-*open set* S,

(3) $\tau_{1,2}$-$cl(A) - A$ *is* $(1,2)^{\star}$-ψ-*closed*,

(4) $A \cup (\tau_{1,2}$-$cl(A))^c$ *is* $(1,2)^{\star}$-ψ-*open.*

Proof. (1) $\Rightarrow$ (2). Let $A \in (1,2)^{\star}$-ψLC*(X). There exist an $(1,2)^{\star}$-ψ-open set S and a $\tau_{1,2}$-closed set G such that $A = S \cap G$. Since $A \subseteq S$ and $A \subseteq \tau_{1,2}$-cl(A), $A \subseteq S \cap$

$\tau_{1,2}$-cl(A). Also since $\tau_{1,2}$-cl(A) $\subseteq$ G, S $\cap$ $\tau_{1,2}$-cl(A) $\subseteq$ S $\cap$ G = A. Therefore A = S $\cap$ $\tau_{1,2}$-cl(A).

(2) $\Rightarrow$ (1). Since S is $(1,2)^{\star}$-ψ-open and $\tau_{1,2}$-cl(A) is a $\tau_{1,2}$-closed set, A = S $\cap$ $\tau_{1,2}$-cl(A) $\in$ $(1,2)^{\star}$-ψLC*(X).

(2) $\Rightarrow$ (3). Since $\tau_{1,2}$-cl(A) $-$ A = $\tau_{1,2}$-cl(A) $\cap$ S^c, $\tau_{1,2}$-cl(A) $-$ A is $(1,2)^{\star}$-ψ-closed by Corollary 2.7.

(3) $\Rightarrow$ (2). Let S = $(\tau_{1,2}$-cl(A) $-$ A$)^c$. Then by assumption S is $(1,2)^{\star}$-ψ-open in X and A = S $\cap$ $\tau_{1,2}$-cl(A).

(3) $\Rightarrow$ (4). Let G = $\tau_{1,2}$-cl(A) $-$ A. Then G^c = A $\cup$ $(\tau_{1,2}$-cl(A)$)^c$ and A $\cup$ $(\tau_{1,2}$-cl(A)$)^c$ is $(1,2)^{\star}$-ψ-open.

(4) $\Rightarrow$ (3). Let S = A $\cup$ $(\tau_{1,2}$-cl(A)$)^c$. Then S^c is $(1,2)^{\star}$-ψ-closed and S^c = $\tau_{1,2}$-cl(A) $-$ A and so $\tau_{1,2}$-cl(A) $-$ A is $(1,2)^{\star}$-ψ-closed.

Theorem 3.41. *Let A be a subset of X. Then A $\in$ $(1,2)^{\star}$-ψLC**(X) if and only if A = S $\cap$ $(1,2)^{\star}$-ψ-cl(A) for some $\tau_{1,2}$- open set S.*

Proof. Let A $\in$ $(1,2)^{\star}$-ψLC**(X). Then A = S $\cap$ G where S is $\tau_{1,2}$-open and G is $(1,2)^{\star}$-ψ-closed. Since A $\subseteq$ G, $(1,2)^{\star}$-ψ-cl(A) $\subseteq$ G. We obtain A = A $\cap$ $(1,2)^{\star}$-ψ-cl(A) = S $\cap$ G $\cap$ $(1,2)^{\star}$-ψ-cl(A) = S $\cap$ $(1,2)^{\star}$-ψ-cl(A).

Converse part is trivial.

Corollary 3.42. *Let A be a subset of X. If A $\in$ $(1,2)^{\star}$-ψLC**(X), then $(1,2)^{\star}$-ψ-cl(A) $-$ A is $(1,2)^{\star}$-ψ-closed and A $\cup$ $((1,2)^{\star}$-ψ-cl(A)$)^c$ is $(1,2)^{\star}$-ψ-open.*

Proof. Let A $\in$ $(1,2)^{\star}$-ψLC**(X). Then by Theorem 3.41, A = S $\cap$ $(1,2)^{\star}$-ψ-cl(A) for some $\tau_{1,2}$-open set S and $(1,2)^{\star}$- ψ-cl(A) $-$ A = $(1,2)^{\star}$-ψ-cl(A) $\cap$ S^c is $(1,2)^{\star}$-ψ-closed in X. If G = $(1,2)^{\star}$-ψ-cl(A) $-$ A, then G^c = A $\cup$ $((1,2)^{\star}$-ψ-cl(A)$)^c$ and G^c is $(1,2)^{\star}$-ψ-open and so is A $\cup$$((1,2)^{\star}$-$\psi$-cl(A)$)^c$.

4. $(1,2)^\star$-ψ-dense sets and $(1,2)^\star$-ψ-submaximal spaces

We introduce the following definition.

Definition 4.1. *A subset A of a space X is called $(1,2)^\star$-ψ-dense if $(1,2)^\star$-ψ-cl(A) $= X$.*

Example 4.2. *Consider the bitopological space X in Example 3.6. Then the set $A = \{b,\ c\}$ is $(1,2)^\star$-ψ-dense in X.*

Recall that a subset A of a space X is called $(1,2)^\star$-dense if $\tau_{1,2}$-cl(A) $= X$.

Proposition 4.3. *Every $(1,2)^\star$-ψ-dense set is $(1,2)^\star$-dense.*

Proof. Let A be an $(1,2)^\star$-ψ-dense set in X. Then $(1,2)^\star$-ψ-cl(A) $=$ X. Since $(1,2)^\star$-ψ-cl(A) $\subseteq \tau_{1,2}$-cl(A), we have $\tau_{1,2}$-cl(A) $=$ X and so A is $(1,2)^\star$-dense.

The converse of Proposition 4.3 need not be true as can be seen from the following example.

Example 4.4. *The set $\{a,\ c\}$ in Example 3.6 is a $(1,2)^\star$-dense in X but it is not $(1,2)^\star$-ψ-dense in X.*

Definition 4.5. *A bitopological space X is called*

1. *$(1,2)^\star$-submaximal if every $(1,2)^\star$-dense subset is $\tau_{1,2}$-open.*
2. *$(1,2)^\star$-$\hat{g}$ (or $(1,2)^\star$-ω)-submaximal if every $(1,2)^\star$-dense subset is $(1,2)^\star$-ω-open.*
3. *$(1,2)^\star$-g-submaximal if every $(1,2)^\star$-dense subset is $(1,2)^\star$-g-open.*
4. *$(1,2)^\star$-rg-submaximal if every $(1,2)^\star$-dense subset is $(1,2)^\star$-rg-open.*

Proposition 4.6. *Let X be a bitopological space.*

1. *If X is $(1,2)^\star$-submaximal, then X is $(1,2)^\star$-$\hat{g}$-submaximal.*

(2) *If X is* $(1,2)^{\star}$-$\hat{g}$*-submaximal, then X is* $(1,2)^{\star}$-*g-submaximal.*

(3) *If X is* $(1,2)^{\star}$-*g-submaximal, then X is* $(1,2)^{\star}$-*rg-submaximal.*

(4) *The respective converses of the above need not be true in general.*

Definition 4.7. *A bitopological space X is called* $(1,2)^{\star}$-ψ*-submaximal if every* $(1,2)^{\star}$-*dense subset in it is* $(1,2)^{\star}$-ψ*-open in X.*

Proposition 4.8. *Every* $(1,2)^{\star}$*-submaximal space is* $(1,2)^{\star}$-ψ*-submaximal.*

Proof. Let X be a $(1,2)^{\star}$-submaximal space and A be a $(1,2)^{\star}$-dense subset of X. Then A is $\tau_{1,2}$-open. But every $\tau_{1,2}$-open set is $(1,2)^{\star}$-ψ-open and so A is $(1,2)^{\star}$-ψ-open. Therefore X is $(1,2)^{\star}$-ψ-submaximal.

The converse of Proposition 4.8 need not be true as can be seen from the following example.

Example 4.9. *For the bitopological space X of Example 3.6, every* $(1,2)^{\star}$*-dense subset is* $(1,2)^{\star}$-ψ*-open and hence X is* $(1,2)^{\star}$-ψ*-submaximal. However, the set* $A = \{a, b\}$ *is* $(1,2)^{\star}$*-dense in X, but it is not* $\tau_{1,2}$*-open in X. Therefore X is not* $(1,2)^{\star}$*-submaximal*

Proposition 4.10. *Every* $(1,2)^{\star}$-ψ*-submaximal space is* $(1,2)^{\star}$-ψ*-submaximal.*

Proof. Let X be an $(1,2)^{\star}$-ψ-submaximal space and A be a $(1,2)^{\star}$-dense subset of X. Then A is $(1,2)^{\star}$-ψ-open. But every $(1,2)^{\star}$-ψ-open set is $(1,2)^{\star}$-ω-open and so A is $(1,2)^{\star}$-ω-open. Therefore is X is $(1,2)^{\star}$-ω-submaximal.

The converse of Proposition 4.10 need not be true as can be seen from the following example.

Example 4.11. *Consider the bitopological space X in Example 3.9. Then X is* $(1,2)^{\star}$-ω*-submaximal but it is not* $(1,2)^{\star}$-ψ*-submaximal, because the set* $A = \{b, c\}$ *is a* $(1,2)^{\star}$*-dense set in X but it is not* $(1,2)^{\star}$-ψ*-open in X.*

Remark 4.12. *From Propositions 3.40, 4.8 and 4.10, we have the following diagram:*

$$(1,2)^*\text{-submaximal} \longrightarrow (1,2)^*\text{-}\psi\text{-submaximal} \longrightarrow (1,2)^*\text{-}\omega\text{-submaximal}$$

$$\downarrow$$

$$(1,2)^*\text{-rg-submaximal} \longleftarrow (1,2)^*\text{-g-submaximal}$$

Theorem 4.13. *A space (X, t) is* $(1,2)^{\star}$*-*ψ*-submaximal if and only if P(X) =* $(1,2)^{\star}$*-*ψ*LC*(X).*

Proof. Necessity. Let A $\in$ P(X) and let V = A $\cup$ ($\tau_{1,2}$-cl(A))^c. This implies that $\tau_{1,2}$-cl(V) = $\tau_{1,2}$-cl(A) $\cup$ ($\tau_{1,2}$-cl(A))^c = X. Hence $\tau_{1,2}$-cl(V) = X. Therefore V is a $(1,2)^{\star}$-dense subset of X. Since X is $(1,2)^{\star}$-ψ-submaximal, V is $(1,2)^{\star}$-ψ-open. Thus A $\cup$ ($\tau_{1,2}$-cl(A))^c is $(1,2)^{\star}$-ψ-open and by Theorem 3.40, we have A $\in$ $(1,2)^{\star}$-ψLC*(X).

Sufficiency. Let A be a $(1,2)^{\star}$-dense subset of X. This implies A $\cup$ ($\tau_{1,2}$-cl(A))^c = A $\cup$ X^c = A $\cup$ $\emptyset$ = A. Now A $\in$ $(1,2)^{\star}$-ψLC*(X) implies that A = A $\cup$ ($\tau_{1,2}$-cl(A))^c is $(1,2)^{\star}$-ψ-open by Theorem 3.40. Hence X is $(1,2)^{\star}$-ψ-submaximal.

Proposition 4.14. *Assume that* $(1,2)^{\star}$*-*ψ*LC(X) forms a topology. For subsets A and B in X, the following are true:*

(1) *If A, B* $\in$ $(1,2)^{\star}$*-*ψ*LC(X), then A* $\cap$ *B* $\in$ $(1,2)^{\star}$*-*ψ*LC(X).*
(2) *If A, B* $\in$ $(1,2)^{\star}$*-*ψ*LC*(X), then A* $\cap$ *B* $\in$ $(1,2)^{\star}$*-*ψ*LC*(X).*
(3) *If A, B* $\in$ $(1,2)^{\star}$*-*ψ*LC**(X), then A* $\cap$ *B* $\in$ $(1,2)^{\star}$*-*ψ*LC**(X).*
(4) *If A, B* $\in$ $(1,2)^{\star}$*-*ψ*LC**(X), then A* $\cap$ *B* $\in$ $(1,2)^{\star}$*-*ψ*LC**(X).*
(5) *If A* $\in$ $(1,2)^{\star}$*-*ψ*LC(X) and B is* $(1,2)^{\star}$*-*ψ*-open (resp.* $(1,2)^{\star}$*-*ψ*-closed), then A* $\cap$ *B* $\in$ $(1,2)^{\star}$*-*ψ*LC(X).*
(6) *If A* $\in$ $(1,2)^{\star}$*-*ψ*LC*(X) and B is* $(1,2)^{\star}$*-*ψ*-open (resp.* $\tau_{1,2}$*-closed), then A* $\cap$ *B* $\in$ $(1,2)^{\star}$*-*ψ*LC*(X).*

(7) *If* $A \in (1,2)^{\star}$-$\psi LC^{**}(X)$ *and* B *is* $(1,2)^{\star}$-ψ-*closed (resp.* $\tau_{1,2}$-*open), then* $A \cap B \in (1,2)^{\star}$-$\psi LC^{**}(X)$.

(8) *If* $A \in (1,2)^{\star}$-$\psi LC^{*}(X)$ *and* B *is* $(1,2)^{\star}$-ψ-*closed, then* $A \cap B \in (1,2)^{\star}$-$\psi LC(X)$.

(9) *If* $A \in (1,2)^{\star}$-$\psi LC^{**}(X)$ *and* B *is* $(1,2)^{\star}$-ψ-*open, then* $A \cap B \in (1,2)^{\star}$-$\psi LC(X)$.

(10) *If* $A \in (1,2)^{\star}$-$\psi LC^{**}(X)$ *and* $B \in (1,2)^{\star}$-$\psi LC^{*}(X)$, *then* $A \cap B \in (1,2)^{\star}$-$\psi LC(X)$.

Proof. By Corollary 2.7 (1) to (8) hold.

(9). Let A = S ∩ G where S is $\tau_{1,2}$-open and G is $(1,2)^{\star}$-ψ-closed and B = P ∩ Q where P is $(1,2)^{\star}$-ψ-open and Q is $\tau_{1,2}$-closed. Then A ∩ B = (S ∩ P) ∩ (G ∩ Q) where S ∩ P is $(1,2)^{\star}$-ψ-open and G ∩ Q is $(1,2)^{\star}$-ψ-closed, by Corollary 2.7. Therefore A ∩ B ∈ $(1,2)^{\star}$-ψLC(X).

Remark 4.15. *Union of two* $(1,2)^{\star}$-ψ-*lc sets (resp.* $(1,2)^{\star}$-ψ-*lc sets,* $(1,2)^{\star}$-ψ-*lc*** *sets) need not be an* $(1,2)^{\star}$-ψ-*lc set (resp.* $(1,2)^{\star}$-ψ-*lc** *set,* $(1,2)^{\star}$-ψ-*lc*** *set) as can be seen from the following examples.*

Example 4.16. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ *and* $\tau_2 = \{\emptyset, X, \{a, b\}\}$. *Then the sets in* $\{\emptyset, X, \{a\}, \{a, b\}\}$ *are called* $\tau_{1,2}$-*open and the sets in* $\{\emptyset, X, \{c\}, \{b, c\}\}$ *are called* $\tau_{1,2}$-*closed. Then* $(1,2)^{\star}$-$\psi LC(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$. *Then the sets* $\{a\}$ *and* $\{c\}$ *are* $(1,2)^{\star}$-ψ-*lc sets, but their union* $\{a, c\} \notin (1,2)^{\star}$-$\psi LC(X)$.

Example 4.17. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{b\}\}$ *and* $\tau_2 = \{\emptyset, X, \{a, b\}\}$. *Then the sets in* $\{\emptyset, X, \{b\}, \{a, b\}\}$ *are called* $\tau_{1,2}$-*open and the sets in* $\{\emptyset, X, \{c\}, \{a, c\}\}$ *are called* $\tau_{1,2}$-*closed. Then* $(1,2)^{\star}$-$\psi LC^{*}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a,$

$c\}$, $X\}$. Then the sets $\{b\}$ and $\{c\}$ are $(1,2)^{\star}$-ψ-lc sets, but their union $\{b,\ c\} \notin$ $(1,2)^{\star}$-ψLC*(X).*

Example 4.18. *Let X= $\{a,\ b,\ c\}$, $\tau_1 = \{\emptyset,\ X,\ \{b\}\}$ and $\tau_2 = \{\emptyset,\ X,\ \{b,\ c\}\}$. Then the sets in $\{\emptyset,\ X,\ \{b\},\ \{b,\ c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset,\ X,\ \{a\},\ \{a,\ c\}\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^{\star}$-ψLC**(X) $= \{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,\ c\},\ \{b,\ c\},\ X\}$. Then the sets $\{a\}$ and $\{b\}$ are $(1,2)^{\star}$-ψ-lc** sets, but their union $\{a,\ b\} \in$ $(1,2)^{\star}$-ψLC**(X).*

We introduce the following definition.

Definition 4.19. *Let A and B be subsets of X. Then A and B are said to be $(1,2)^{\star}$-ψ-separated if $A \cap (1,2)^{\star}$-ψ-cl(B) $= \emptyset$ and $(1,2)^{\star}$-ψ-cl(A) $\cap B = \emptyset$.*

Example 4.20. *For the bitopological space X of Example 3.6. Let $A = \{b\}$ and let $B = \{c\}$. Then $(1,2)^{\star}$-ψ-cl(A) $= \{a,\ b\}$ and $(1,2)^{\star}$-ψ-cl(B) $= \{a,\ c\}$ and so the sets A and B are $(1,2)^{\star}$-ψ-separated.*

Proposition 4.21. *For a bitopological space X, the followings are true:*

(1) *Let A, $B \in (1,2)^{\star}$-ψLC(X). If A and B are $(1,2)^{\star}$-ψ-separated then $A \cup B \in$ $(1,2)^{\star}$-ψLC(X).*
(2) *Let A, $B \in (1,2)^{\star}$-ψLC*(X). If A and B are separated (i.e., $A \cap \tau_{1,2}$-cl(B) $=$ $\emptyset$ and $\tau_{1,2}$-cl(A) $\cap B = \emptyset$), then $A \cup B \in (1,2)^{\star}$-$\psi LC$*(X).*
(3) *Let A, $B \in (1,2)^{\star}$-ψLC**(X). If A and B are $(1,2)^{\star}$-ψ-separated then $A \cup B$ $\in (1,2)^{\star}$-ψLC**(X).*

Proof. (1) Since A, B $\in (1,2)^{\star}$-ψLC(X), by Theorem 3.39, there exist $(1,2)^{\star}$-ψ-open sets U and V of X such that A $=$ U $\cap$ $(1,2)^{\star}$-ψ-cl(A) and B $=$ V $\cap$ $(1,2)^{\star}$-ψ-cl(B).

Now G = U ∩ (X − $(1,2)^{\star}$-ψ-cl(B)) and H = V ∩ (X − $(1,2)^{\star}$-ψ-cl(A)) are $(1,2)^{\star}$-ψ-open subsets of X. Since A ∩ $(1,2)^{\star}$-ψ-cl(B) = ∅, A ⊆ $((1,2)^{\star}$-ψ-cl(B)$)^c$. Now A = U ∩ $(1,2)^{\star}$-ψ-cl(A) becomes A ∩ $((1,2)^{\star}$-ψ-cl(B)$)^c$ = G ∩ $(1,2)^{\star}$-ψ-cl(A). Then A = G ∩ $(1,2)^{\star}$-ψ-cl(A). Similarly B = H ∩ $(1,2)^{\star}$-ψ-cl(B). Moreover G ∩ $(1,2)^{\star}$-ψ-cl(B) = ∅ and H ∩ $(1,2)^{\star}$-ψ-cl(A) = ∅. Since G and H are $(1,2)^{\star}$-ψ-open sets of X, G ∪ H is $(1,2)^{\star}$-ψ-open. Therefore A ∪ B = (G ∪ H) ∩ $(1,2)^{\star}$-ψ-cl(A ∪ B) and hence A ∪ B ∈ $(1,2)^{\star}$-ψLC(X).

(2) and (3) are similar to (1), using Theorems 3.40 and 3.41.

Remark 4.22. *The assumption that A and B are $(1,2)^{\star}$-ψ-separated in (1) of Proposition 4.21 cannot be removed. In the bitopological space X in Example 4.16, the sets $\{a\}$ and $\{c\}$ are not $(1,2)^{\star}$-ψ-separated and their union $\{a, c\} \notin (1,2)^{\star}$-$\psi$LC(X).*

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