Generalized Sasakian-Space-Forms admitting Quarter–Symmetric Metric Connection

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Abstract: *The object of the present paper is to study generalized Sasakian-space-forms admitting quarter-symmetric metric connection. The relation between the curvature tensors of quarter-symmetric metric connection and linear connection has been obtained. Also, the properties of projective and conformal curvature tensors of quarter-symmetric metric connection on a generalized Sasakian-space–form have been studied.*

Key-Words: *generalized Sasakian-space-form, quarter-symmetric metric connection, projective curvature tensor and conformal curvature tensor,η-Einstein manifold.*

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1. Introduction: In 1975, Golab[14] defined and studied quarter-symmetric connection in a differentiable manifold.A linear connection $\overline{\nabla}$on an n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric-symmetric connection if its torsion tensor $\overline{\nabla}$ defined by

T(X,Y) =$\overline{\nabla}_X Y$-$\overline{\nabla}_Y X$-[X,Y]

is of the form

$\overline{\nabla}(X,Y)$=$\eta(Y)\phi X$-$\eta(X)\phi Y$,   (1.1)

where η is 1-form and ϕ is a tensor of type (1,1).In addition,a quarter-symmetric linear connection $\overline{\nabla}$ satisfies the condition.

$(\overline{\nabla}_X g)(Y,Z)=0$,   (1.2)

for all X, Y, Z∈TM, where TM is the Lie algebra of vector fields of the manifoldM, then $\overline{\nabla}$is said to be quarter-symmetric metric connection. In particular,ifφX=XandφY=Y, then the quarter-symmetric connection reduces to asemi-symmetric connection [14].Quarter-symmetric metric connection are also studied by Biswas and De [7], De and De [9] De and Mondal [10], Singh, Pandey and Tiwari [17], Yano and Imai [22] and manyothers.

On the other hand, a generalized Sasakian-space-form was defined by Alegreeet al. [1] as the almost contact manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3, \quad (1.3)$$

where f_1, f_2, f_3 are some differential function on $M^{2n+1}$and

$R_1(X,Y)Z$=g(Y, Z)X-g(X, Z)Y,

$$R_2(X,Y)Z = g(X,\phi Y) - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \quad (1.4)$$

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$

for any vector fields X, Y, Z on M^{2n+1}. In such a case we denote the manifold as M (f_1, f_2, f_3). This kind of manifold appears as a generalization of the well-known Sasakian-space-forms by taking $f_1 = (c-1)/4$.It is known that any three-dimensional (α, β)-trans-Sasakian manifold with α, β depending on ξ is a generalized Sasakian-space-form [2].Alegre et al. give results in [4] about B. Y. Chen's inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. Al-Ghefari et al. analyse the CR submanifolds of generalized Sasakian-space-forms ([5],[6]). In [15], Kim studied conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms. De and Sarkar [11] have studied generalized Sasakian-space-forms regarding conharmonic curvature tensor.Conharmoniccurvature tensor of generalized Sasakian-Space-forms have also been studied by De, Singh and Pandey [12]. Singh and Pandey ([16], [18]) have studied generalized Sasakian-space-forms and many others.

2. Generalised Sasakian-Space-Form:

In an almost contact metric manifold, we have [8]

$$\phi^2 X = -X + \eta(X)\xi, \phi\xi=0, \qquad (2.1)$$

$$\eta(\xi)=1,\ g(X,\xi) =\eta(X),\ \ \eta(\phi X)=0, \qquad (2.2)$$

$$g(\phi X, \phi Y)= g(X,Y) - \eta(X)\eta(Y), \qquad (2.3)$$

$$g(\phi X, Y) = -g(X, \phi Y),\ \ g(\phi X, X) =0, \qquad (2.4)$$

$$(\nabla_X \eta)(Y) =g(\nabla_X \xi, Y), \qquad (2.5)$$

where ϕ is a (1, 1) tensor, ξ is a vector field, η is a 1-form and g is a Riemannian metric. The metric g induces an inner product on the tangent space of the manifold. Again, we know that [1] in a generalized Sasakian-space-form

$$R(X, Y)Z =f_1[g(Y, Z)X-g(X, Z)Y]$$
$$+f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z]$$
$$+f_3[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \qquad (2.6)$$

for any vector fields X,Y,Z on M, where R denotes the curvature tensor of M and f_1, f_2, f_3 are smooth functions on the manifold. The Ricci operator Q, Ricci tensor S and the scalar curvature r of the manifold of dimension (2n+1) are respectively given by [15]

$$QX= (2nf_1 + 3f_2 - f_3)X-(3f_2 + (2n-1)f_3)\eta(X)\xi, \qquad (2.7)$$

$$S(X ,Y)=(2nf_1 + 3f_2 - f_3)g(X,Y) -(3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \qquad (2.8)$$

$$r=2n(2n+1)f_1 + 6nf_2-4nf_3. \qquad (2.9)$$

In view of equations (2.6), (2.7) and (2.8), we have

$$R(X, Y)\xi =(f_1 - f_3)[\eta(Y)X-\eta(X)Y], \qquad (2.10)$$

$$R(\xi, X)Y= (f_1 - f_3)[g(X,Y)\xi-\eta(Y)X], \qquad (2.11)$$

$$\eta(R(X,Y)Z)=(f_1 - f_3)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)], \qquad (2.12)$$

$$S(X, \xi) =2n\,(f_1 - f_3)\eta(X). \qquad (2.13)$$

3. Quarter-Symmetric Metric Connection

Let $\overline{\nabla}$ be the linear connection and ∇ be the Levi-Civita connection of a generalized Sasakian space-formM^n such that

$\overline{\nabla}_X Y = \nabla_X Y + H(X,Y)$, (3.1)

where H is a tensor field of type (1,1). For $\overline{\nabla}$ to be a quarter-symmetric metric connection in M^n, we have

H(X,Y) $=\frac{1}{2}[\overline{T}(X,Y) + \overline{T}'(X,Y) + \overline{T}$ '(Y, X)] (3.2)

and

g(T' (X,Y),Z)= g($\overline{T}'(Z,X),Y$). (3.3)

In view of equation (1.1) and (3.3), we get

$\overline{T}'(X,Y) = \eta(X)\phi Y - g(\phi X,Y)\xi$, (3.4)

Now, using equations (1.1) and (3.4) in equation (3.2), we get

H(X,Y) = $\eta(Y)\phi X - g(\phi X,Y)\xi$.(3.5)

Hence a quarter- symmetric metric connection $\overline{\nabla}$ in a generalized Sasakian space form M^n given by

$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi$.(3.6)

Thus, the above equation is the relation between quarter-symmetric metric connection and

the Levi–Civita connection.The curvature tensor $\bar{R}$ of M^n with respect to quarter–symmetric metric connection$\overline{\nabla}$ is defined by

$\bar{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z$. (3.7)

In view of equation (3.6), above equation takes the form

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X$$

$+(f_1 - f_3)[\{\eta(X)Y - \eta(Y)X\}\eta(Z) + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi]$. (3.8)

where $\bar{R}$and R are the curvature tensors of M^n with respect to $\overline{\nabla}$ and ∇ respectively.

From equation (3.8),it follows that

$$\bar{R}(X,Y,Z,U) = R(X,Y,Z,U) + g(\phi X,Z)g(\phi Y,U) - g(\phi Y,Z)g(\phi X,U)$$
$$+(f_1 - f_3)[\{\eta(X)g(U,Y) - \eta(Y)g(U,X)\}\eta(Z)$$
$$+\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(U). \quad (3.9)$$

Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of the tangent space at each point of the manifold.Putting X=U=$e_i$in equation (3.9) and summing over i,1$\leq i \leq n$, we get

$$\bar{S}(Y,Z) = S(Y,Z) - (1+f_1-f_3)g(Y,Z) + (1-f_1+f_3)\eta(Y)\eta(Z), \quad (3.10)$$

which gives

$$\bar{Q}Y = QY - (1+f_1-f_3)Y + (1-f_1+f_3)\eta(Y)\xi. \quad (3.11)$$

where $\bar{Q}$ and Q are the Ricci operators of type (1.1),i.e.S(Y,Z)=g($\bar{Q}Y,Z$) and S(Y,Z)=g(QY,Z) with respect to $\bar{\nabla}$and ∇ respectively.

Again, putting Y=Z=e_i in equation (3.10), we get

$\bar{r}$=r- (n-1) - (n+1) (f_1-f_3),(3.12)

where $\bar{r}$ and r are the scalar curvature with respect to $\bar{\nabla}$and ∇respectively.

Now, writing two more equations by the cyclic permutations of X, Y and Z, from equation (3.8), we get

$$\bar{R}(Y,Z)X = R(Y,Z)X + g(\phi Y,X)\phi Z - g(\phi Z,X)\phi Y$$

$$+(f_1-f_3)[\{\eta(Y)Z-\eta(Z)Y\}\eta(X) + \{g(Y,X)\eta(Z) - g(Z,X)\eta(Y)\}\xi]. \quad (3.13)$$

and

$$\bar{R}(Z,X)Y = R(Z,X)Y + g(\phi Z,Y)\phi X - g(\phi X,Y)\phi Z$$

$$+(f_1-f_3)[\{\eta(Z)X-\eta(X)Z\}\eta(Y) + \{g(Z,Y)\eta(X) - g(X,Y)\eta(Z)\}\xi]. \quad (3.14)$$

Adding all these three equations (3.8), (3.13) and (3.14)and using the fact that

$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$=0, we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 2[g(\phi X,Y)\phi Z + g(\phi Y,X)\phi Z$$

$$+g(\phi Z,Y)\phi X)]. \quad (3.15)$$

Thus, we can state as follows-

Theorem (3.1): Generalized Sasakian-space- form admitting quarter symmetric metric connection satisfies equation (3.15).

Now, interchanging X and Y in equation (3.9),we get

$$\bar{R}(Y,X,Z,U) = R(Y,X,Z,U) + g(\phi Y,Z)g(\phi X,U) - g(\phi X,Z)g(\phi Y,U)$$

$$+(f_1-f_3)[\{\eta(X)g(U,X) - \eta(X)g(U,X)\}\eta(Z)$$

$+\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(U)]$. (3.16)

Adding above equation with equation (3.9) with the fact that $R(X,Y,Z,U) + R(Y,X,Z,U) = 0$, we get

$\bar{R}(X,Y,Z,U) + \bar{R}(Y,X,Z,U) = 0$. (3.17)

Again interchanging Z and U in equation (3.9) and addingto equation (3.9) with fact that $R(X,Y,Z,U) + R(X,Y,U,Z) = 0$, we get

$\bar{R}(X,Y,Z,U) + \bar{R}(X,Y,U,Z) = 0$. (3.18)

Now, interchanging pair of slots in equation (3.9) and subtracting these equations (3.9) with fact that $R(X,Y,Z,U) = R(Z,U,X,Y)$, we get

$\bar{R}(X,Y,Z,U) = \bar{R}(Z,U,X,Y)$.(3.19)

Thus, in view of equations (3.17), (3.18) and (3.19),we can state as follows.

Theorem (3.2): In a generalized Sasakian space-form admitting quarter-symmetric metric connection, we have

(i) $\bar{R}(X,Y,Z,U) + \bar{R}(Y,X,Z,U) = 0$.
(ii) $\bar{R}(X,Y,Z,U) + \bar{R}(X,Y,U,Z) = 0$.
(iii) $\bar{R}(X,Y,Z,U) - \bar{R}(Z,U,X,Y) = 0$.

Now, putting Z=ξ in equation (3.8) and using equations (2.2) and (2.4), we get

$\bar{R}$(X,Y)ξ=0.(3.20)

Taking the inner product of equation (3.8) with ξ and using equations(2.2) and(2.13), we get

$\eta(\bar{R}(X,Y)Z) = 0$. (3.21)

PuttingX=ξ, in equation (3.8) and using equations (2.1),(2.13), we get

$\bar{R}(\xi,Y)Z = 0$. (3.22)

Again

$\bar{R}(\xi,Y)Z = -\bar{R}(Y,\xi)Z = 0$. (3.23)

By putting Y=ξ in equation (3.10) and use of equation (2.13),we obtain

$\bar{S}$ (ξ, Z) =(2n-2)$(f_1 - f_3)\eta(Z)$. (3.24)

Thus, by virtue of equations (3.20), (3.21), (3.22), (3.23) and (3.24) we can state follows.

Theorem (3.3): In ageneralized Sasakian–space-form admitting quarter-symmetric metric connection, we have

(i) $\bar{R}$(X,Y)ξ=0,

(ii) $\eta(\bar{R}(X,Y)Z) = 0,$

$(iii)\bar{R}(\xi, Y)Z = 0,$

(iv)$\bar{R}(\xi, Y)Z = -\bar{R}(Y, \xi)Z = 0,$

$(v)\bar{S}$ (ξ, Z) =(2n-2)$(f_1 - f_3)\eta(Z)$.

Now, consider

$$(R(\xi, X).\overline{R})(Y,Z)U=0, \qquad (3.25)$$

which gives

$$R(\xi, X).\bar{R}(Y,Z)U-\bar{R}(R(\xi, X)Y,Z)U-\bar{R}(Y, R(\xi, X)Z)U-\bar{R}(Y,Z)R(\xi,X)U = 0. \qquad (3.26)$$

In view of equations (2.11),(3.20) and (3.21),above equation takes the form

$$(f_1 - f_3)[g(X,\bar{R}(Y,\xi)U)+\eta(Y)\bar{R}(X,Z)U + \eta(Z)\bar{R}(Y,X)U$$
$$-g(X,U)\bar{R}(Y,Z)\xi+\eta(U)\bar{R}(Y,Z)X] = 0. \quad (3.27)$$

By virtue of equation (3.8), above equation reduces to

$$(f_1 - f_3)[g(X, R(Y, Z, U)\xi + \eta(Y)R(X, Z)U + \eta(Z)R(Y, X) - g(X, U)R(Y,Z)\xi$$
$$+\eta(U)R(Y,Z)X + g(\phi Y, U)g(X, \phi Z)\xi - g(\phi Z, U)g(X, \phi Y)\xi$$
$$+ g(\phi X, U)\eta(Y)\phi Z - g(\phi Z, U)\eta(Y)\phi X + g(\phi Y, U)\eta(Z)\phi X$$
$$-g(\phi X, U)\eta(Z)\phi Y - g(X, U)\eta(\phi Y)\phi Z + g(X, U)\eta(\phi Z)\phi Y$$
$$+ g(\phi Y, X)\eta(U)\phi Z - g(\phi Z, X)\eta(U)\phi Y$$
$$+(f_1 - f_3)[2g(Y,U)\eta(X)\eta(Z)\xi - g(Z,U)\eta(X)\eta(Y)\xi$$
$$+2\eta(X)\eta(Y)\eta(U)Z-2\eta(X)\eta(Y)\eta(Z)Y - 2g(X,U)\eta(Y)\eta(Z)\xi$$
$$-g(X,U)\eta(Y)Z + g(X,U)\eta(Z)Y - g(X,U)\eta(Y)\eta(Z)\xi] = 0. \quad (3.28)$$

Taking the inner product of above equation with ξ and using equations (2.2) and (2.12), we get

$$(f_1 - f_3)[R(Y, Z, U)X+(f_1 - f_3)\{g(Z,X)\eta(U)\eta(Y) + g(Y,U)\eta(X)\eta(Z)$$
$$-3g(X,U)\eta(Y)\eta(Z) + g(\phi Y,U)g(X,\phi Z) - g(\phi Z,U)g(X,\phi Y)] = 0. \qquad (3.29)$$

Putting Y=X=e_i in above equation and taking summation over i,1≤ $i \leq n$, we get

$$S(Z, U)=g(U, Z)+[(f_1 - f_3)(n+1)-1]\eta(U)\eta(Z). \qquad (3.30)$$

Thus, we can state as follows-

Theorem (3.4): A generalized Sasakian space-form with quarter-symmetric metric connection satisfying $(R(\xi, X).\overline{R)}\,(Y, Z)U = 0$, is an η-Einstein manifold.

4. Projective Curvature Tensor of Generalized Sasakian-Space Forms admitting Quarter-Symmetric Metric Connection:

Projective curvature tensor of quarter–symmetric metric connection $\overline{\nabla}$ in M^n is defined as

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y], \quad (4.1)$$

which on using equations (3.8) and (3.10), gives

$$\bar{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X$$

$$-\left[\frac{(n-1)(f_1-f_3)+1)}{(n-1)}\right][\eta(X)Y - \eta(Y)X]\eta(Z)$$

$$+\left[\frac{(1+3f_1-f_3)}{(n-1)}\right][g(Y,Z)X - g(X,Z)Y], \quad (4.2)$$

which gives

$$\bar{P}(X,Y)Z = P(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X$$

$$-\left[\frac{(n-1)(f_1-f_3)+1)}{(n-1)}\right][\eta(X)Y - \eta(Y)X]\eta(Z)$$

$$+\left[\frac{(1+3f_1-f_3)}{(n-1)}\right][g(Y,Z)X - g(X,Z)Y], \quad (4.3)$$

where P(X, Y) Z is the projective curvature tensor [11] of connection ∇ defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \quad (4.4)$$

Putting X=ξ in equation (4.2) and using equations (2.1), (2.2), (2.6), (2.11) and (2.13), we get

$$\bar{P}(\xi,Y)Z = \left[\frac{(n-2)(f_1-f_3)-1)}{n-1}\right][g(Y,Z)\xi - \eta(Z)Y] + \left[\frac{(3n-2)(f_1-f_3)+1)}{n-1}\right]\eta(Z)Y$$

$$+\left[\frac{(3f_2+(3n-3)f_3-(n-2)f_1)-1)}{(n-1)}\right]\eta(Y)\eta(Z)\xi$$

$$-\left[\frac{(2nf_1+3f_2-f_3)}{n-1}\right]g(Y,Z)\xi. (4.5)$$

Againputting Z=ξ, in equation (4.2) and using (2.1), (2.2), (2.10) and (2.13), we get

$$\bar{P}(X,Y)\xi = -\left[\frac{2n(f_1-f_3)+2}{n-1}\right][\eta(Y)X - \eta(X)Y]. \quad (4.6)$$

Now, taking the inner product of equation (4.1) with ξ and using equations (2.1), (2.2), (2.4),(2.8),(3.10)and (3.21), we get

$\eta(\bar{P}(X,Y)Z)$=$-\frac{1}{(n-1)}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]$

$-\left[\frac{(1-f_1+f_3)}{n-1}\right][g(X,Z)\eta(Y)-g(Y,Z)\eta(X)],$ (4.7)

which gives

$\eta(\bar{P}(X,Y)Z)$=$\left[\frac{(2n+1)f_1-3f_2-2f_3-1}{(n-1)}\right][g(X,Z)\eta(Y)-g(Y,Z)\eta(X)].$ (4.8)

Thus in view of equations (4.5),(4.6) and (4.7), we can state the follows-

Theorem (4.1): In a generalized Sasakian-space–form with quarter–symmetric metric connection, we have

(i) $\bar{P}(\xi,Y)Z=\left[\frac{(n-2)(f_1-f_3)-1)}{n-1}\right]$[g(Y,Z)$\xi$- η(Z)Y] + $\left[\frac{(3n-2)(f_1-f_3)+1)}{n-1}\right]\eta$(Z)Y

+[$\frac{(3f_2+(3n-3)f_3-(n-2)f_1)-1)}{(n-1)}$]$\eta(Y)\eta(Z)\xi$

-[$\frac{(2nf_1+3f_2-f_3)}{n-1}$]g(Y, Z)$\xi$.

(ii) $\bar{P}(X,Y)\xi=-[\frac{2n(f_1-f_3)+2}{n-1}][\eta(Y)X-\eta(X)Y]$..

(iii) $\eta(\bar{P}(X,Y)Z)$=$-\frac{1}{(n-1)}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]$

$$-\left[\frac{(1-f_1+f_3)}{n-1}\right][g(X,Z)\eta(Y)-g(Y,Z)\eta(X)].$$

Now, interchanging X and Y in equation (4.2) and adding these equation to equation (4.2) with the fact that $R(X,Y)Z+R(Y,Z)X=0,$ we get

$\bar{P}(X,Y)Z+\bar{P}(Y,X)Z=0.$ (4.9)

Again from equation (4.2) writing two more equations by the cyclic permutations of X, Y and Z and adding all these three equations with the fact that $R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0,$ we get

$\bar{P}(X,Y)Z+\bar{P}(Y,Z)X+\bar{P}(Z,X)Y=0.$ (4.10)

Theorem (3.2): In a generalized Sasakian space-form admitting quarter-symmetric metric connection, we have

$$(i)\bar{P}(X,Y)Z+\bar{P}(Y,X)Z=0.$$

(ii) $\bar{P}(X,Y)Z+\bar{P}(Y,Z)X+\bar{P}(Z,X)Y=0.$

Now, suppose

$(R(\xi, X).\overline{P})(Y,Z)U=0,$ (4.11)

which gives

$R(\xi, X).\bar{P}(Y,Z)U-\bar{P}(R(\xi, X)Y,Z)U-\bar{P}(Y, R(\xi, X)Z)U-\bar{P}(Y,Z)R(\xi, X)U = 0.$ (4.12)

By virtue of equations (2.11) above equation takes the form

$$(f_1 - f_3)[g(X,\bar{P}(Y,\xi)U)\xi-\eta(\bar{P}(Y,Z)U)X-g(X, Y)\bar{P}(\xi,Z)U + \eta(Y)\bar{P}(X,Z)U$$
$$-g(X,Z)\bar{P}(Y,\xi)U + \eta(Z)\bar{P}(Y,X)U$$
$$-g(X,U)\bar{P}(Y,Z)\xi+\eta(U)\bar{P}(Y,Z)X] = 0. \quad (4.13)$$

Taking the inner product of above equation with ξ and using equation (2.2), we get

$$(f_1 - f_3)[g(X,\bar{P}(Y,\xi)U)-\eta(\bar{P}(Y,Z)U)\eta(X)-g(X, Y)\eta(\bar{P}(\xi,Z)U) + \eta(Y)\eta(\bar{P}(X,Z)U)$$
$$-g(X,Z)\eta(\bar{P}(Y,\xi)U) + \eta(Z)\eta(\bar{P}(Y,X)U)$$
$$-g(X,U)\eta(\bar{P}(Y,Z)\xi)+\eta(U)\eta(\bar{P}(Y,Z)X)] = 0. \quad (4.14)$$

In view of equations (4.2), (4.2), (4.5), (4.6) and (4.7) above equation takes the form

$$(f_1 - f_3)[g(X, R(Y, Z, U) - \frac{1}{n-1}\{S(Z, U)g(X, Y) - S(Y, Z)\}+g(\phi Y, U)g(X, \phi Z)$$
$$-g(\phi Z, U)g(X, \phi Y) + \left[\frac{(n-2)(f_1 - f_3) - 1}{n-1}\right][g(X,Z)\eta(Y) - g(X,U)\eta(Z)]$$
$$-[\frac{(1+f_1-f_3)}{n-1}][g(Z, U)g(X, Y)-g(Y, U)\, g(X, Z)]+[\frac{(2n+1)f_1+3f_2-2f_3-1}{n-1}][g(Z, U)g(X, U)$$
$$+g(X, U)\eta(Y)\eta(Z) + g(X,Z)g(Y,U) - g(X,Y)\eta(U)\eta(Z)]] = 0. \quad (4.15)$$

Putting $Y=X=e_i$ in above equation and taking summation over $i, 1\leq i \leq n$, we get

$$S(Z, U)=[\frac{2(n-1)+(4n-2n^2)f_1+3(n+1)f_2-(3n+1)f_3}{n-1}]\, g(U, Z)$$
$$-[(2n+1)f_1 + 3f_2 - 2f_3 - 1)(n + 1) - 2]\eta(U)\eta(Z). \quad (4.16)$$

Thus, we can state as follows-

Theorem (3.4): Ageneralized Sasakian space-forms with quarter-symmetric metric connection satisfying $(R(\xi, X).\overline{R}\,)(Y,Z)U = 0$, is an η-Einstein manifold.

5. Conformal Curvature Tensor of Generalized Sasakian-Space –Form admitting Quarter-Symmetric Metric Connection:

Conformal curvature tensor $\bar{C}$ of quarter-symmetric metric connection $\bar{\nabla}$ in M^n is defined as

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\overline{QX} - g(X,Z)\overline{QY}]$$

$+\frac{\bar{r}}{(n-1)(n-2)}$ [g(Y,Z)X-g(X,Z)Y]. (5.1)

Using equations (3.8),(3.10),(3.11) and (3.12), we get

$$\bar{C}(X,Y)Z = R(X,Y)Z - \left[\frac{1}{n-2}\right][S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

$+\frac{r}{(n-1)(n-2)}$[g(Y,Z)X-g(X, Z)Y]+g$(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X$

$$+\left[(f_1-f_3)+\frac{(1-f_1+f_3)}{(n-2)}\right]\{\eta(Y)X-\eta(X)Y\}\eta(Z)$$

$$+\left[(f_1-f_3)+\frac{(1-f_1+f_3)}{(n-2)}\right][g(X,Z)\eta(X)-g(Y,Z)\eta(Y)]\xi$$

$$+\left[\frac{(1-f_1+f_3)-n(1+f_1-f_3)}{(n-1)(n-2)}\right][g(Y,Z)X-g(X,Z)Y], (5.2)$$

which yields

$\bar{C}(X,Y)Z = C(X,Y)Z$ +g$(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X$

$$+\left[\frac{(1-f_1+f_3)-(n-2)(f_1-f_3)}{(n-2)}\right][\eta(Y)X-\eta(X)Y]\eta(Z)$$

$$+\left[\frac{(f_1-f_3)(n-2)-(1-f_1+f_3)}{(n-2)}\right][g(X,Z)\eta(X)-g(Y,Z)\eta(Y)]\xi$$

$$+\left[\frac{(1-f_1+f_3)-n(1+f_1-f_3)}{(n-1)(n-2)}\right][g(Y,Z)X-g(X,Z)Y], \qquad (5.3)$$

where C(X, Y)Z is the conformal curvature tensor of connection ∇ in M^n [8] defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

$+\frac{r}{(n-1)(n-2)}$ [g(Y,Z)X-g(X,Z)Y]. (5.4)

Now, interchanging X and Y in equation (5.2) and adding these equation to equation (5.2) with the fact that $R(X,Y)Z + R(Y,Z)X = 0,$ we get

$\bar{C}(X,Y)Z + \bar{C}(Y,X)Z = 0.$ (5.5)

Again from equation (5.2) writing two more equations by the cyclic permutations of X, Y and Z and adding all these three equations with the fact that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$ we get

$$\bar{C}(X,Y)Z + \bar{C}(Y,Z)X + \bar{C}(Z,X)Y = 0. \qquad (5.6)$$

Thus, we can state as follows

Theorem (3.2): In a generalized Sasakian space-form admitting quarter-symmetric metric connection, we have

(i) $\bar{C}(X,Y)Z + \bar{C}(Y,X)Z = 0.$

(ii) $\bar{C}(X,Y)Z + \bar{C}(Y,Z)X + \bar{C}(Z,X)Y = 0.$

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