Redefined N-subgroups of a near-ring

Pradip Saikia [#1], Gopi Kanta Barthakur [*2]
[#]Research Scholar,Department of Mathematics, Gauhati University,India
[*]Assistant lecturer, Ghana Kanta Baruah college, Morigaon, Assam, India

Abstract: *In this paper we introduce basic notion of* $(\overline{\in},\overline{\in}\vee\overline{q})$ *fuzzy N-subgroup of a near-ring N and introduce the basic results and properties of* $(\overline{\in},\overline{\in}\vee\overline{q})$ *fuzzy N-subgroup E. We also discuss* $(\overline{\in},\overline{\in}\vee\overline{q})$ *fuzzy ideal of near-ring group E towards the end of this paper.*

Keywords: *near-ring, N-subgroups,* $(\overline{\in},\overline{\in}\vee\overline{q})$ *fuzzy N-subgroup,* $(\overline{\in},\overline{\in}\vee\overline{q})$ *fuzzy ideal*

1. Introduction:

After the introduction of the concept of fuzzy set by Zadeh [1] in 1965, these ideas have been applied to various algebraic structures like group, rings, modules and so on. In 1971 Rosenfield [2] defined fuzzy subgroups and gave some of its properties. Since then fuzzy algebraic substructures has been pursued to many directions of near-ring also. The notion of fuzzy sub-ring and ideals were introduced by Wang-Jin Liu [3] in 1982. Later S. Abou Zaid [4] introduced the notion of a fuzzy sub near-ring and fuzzy ideals of near-ring. Moreover Helen K. Saikia and Lila K. Barthakur [5] also gave notion about fuzzy N-subgroups and fuzzy ideals. Kim and Jun considered the fuzzification of N-subgroups in a near-ring N. The concept of fuzzy point and its belongingness to and quasi-coincidence was introduced by Ming and Ming [6],which was used by Bhakat and Das [7,8] to introduce the notion of $(\in,\in\vee q)$-fuzzy subgroups and notion of $(\in,\in\vee q)$-fuzzy subrings. There after Zhan Jian-ming and Davvaz B. [9] introduce the notion of $(\overline{\in},\overline{\in}\vee\overline{q})$ fuzzy sub near-ring (ideals) of a near-ring. Luo and Zhan [10] later gave the notion of $(\in,\in\vee q)$ fuzzy R-subgroups of near-ring. In this paper we introduce the notion of $(\overline{\in},\overline{\in}\vee\overline{q})$ fuzzy N-subgroups and $(\overline{\in},\overline{\in}\vee\overline{q})$ fuzzy ideals of an N-group E.

2. Preliminaries:

In this section we will recall some basic notions.

Definition2.1: A near-ring N is a system with two binary operations + and . such that

(i) (N,+) is a group.(not necessarily abelian)
(ii) (N,.) is a semi group
(iii) $x.(y+z)=x.y+x.z$ for all $x,y,z\in N$

In a near-ring only one distributive law holds.

Definition 2.2: Let N be a near-ring and E an additive group. E is said to be a near-ring group or an N-group if there exists a mapping $N\times E\to E,(n,e)\to ne$ such that

(i) $(n+m)e=ne+me$

(ii) $(nm)e=n(me)$

(iii) $1e=e$ for all $n,m\in N,e\in E$

Definition 2.3: A normal subgroup A of E is called an ideal of E if $n(a+e)-ne\in A$ for all $n\in N,a\in A,e\in E.$

Definition 2.4: Let μ be a fuzzy subset of an N-group E. Then μ is called a fuzzy N-subgroup of E if for all $n\in N,x,y\in E$,

(i) $\mu(x+y)\geq\mu(x)\wedge\mu(y)$
(ii) $\mu(-x)\geq\mu(x)$
(iii) $\mu(nx)\geq\mu(x)$

Definition 2.5: Let μ be a fuzzy subset of an N-group E . The μ is said to be a fuzzy ideal of E if for all $a,x,y\in E,n\in N$,

(i) $\mu(x+y)\geq\mu(x)\wedge\mu(y)$
(ii) $\mu(-x)\geq\mu(x)$
(iii) $\mu(x+y-x)\geq\mu(y)$
(iv) $\mu[n(a+x)-na]\geq\mu(x)$

Definition 2.6: Let $A \subseteq E$. Then the function $\mu : E \rightarrow [0,1]$ defined by

$$\mu(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

is a fuzzy subset of E which is the characteristic function χ_A of A.

Definition 2.7: A fuzzy subset μ of E of the form

$$\mu(y) = \begin{cases} t(\neq 0), y = x \\ 0, y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t

Definition2.8: A fuzzy point x_t is said to belong to (resp. be quasi-coincidence with) a fuzzy subset μ, written as $x_t \in \mu$ (resp. $x_t q\mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q\mu$ then we write $x_t \in \vee q\mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$) then we call $x_t \overline{\in} \mu$ (resp., $x_t \overline{q}\mu$)

Theorem 2.9: A non empty subset A of E is an N-subgroup of E if and only if the characteristic function χ_A of A is a fuzzy N-subgroup of E.

Theorem 2.10: Let A be a non empty subset of E. Then A is an ideal of E if and only if the characteristic function χ_A of A is a fuzzy ideal of E.

3. Main results:

Definition 3.1: A fuzzy subset μ of E is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$ fuzzy N-subgroup of E if for all $t, r \in (0,1], n \in N, x, y \in E$ the following holds:

(F1a) $(x+y)_{t \wedge r} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$

(F1a') $(-x)_t \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$

(F1b) $(nx)_t$ implies $x_t \overline{\in} \vee \overline{q}\mu$

Theorem 3.2: A fuzzy subset μ of E is an $(\overline{\in}\ \overline{\in} \vee \overline{q})$ fuzzy N-subgroup of E if and only if for all $n \in N, x, y \in E$

(F2a) $\mu(x+y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$

(F2a') $\mu(-x) \vee 0.5 \geq \mu(x)$

(F2b) $\mu(nx) \vee 0.5 \geq \mu(x)$

Proof : We prove (F1a) $\Leftrightarrow$ (F2a) .

(F1a) $\Rightarrow$ (F2a): If there exists $x, y \in E$ such that $\mu(x+y) \vee 0.5 < t = \mu(x) \wedge \mu(y)$ then $0.5 < t \leq 1, (x+y)_t \overline{\in} \mu$ but $x_t \in \mu, y_r \in \mu$. By (F1a) we have $x_t \overline{q}\mu$ or $y_t \overline{q}\mu$. Then $(t \leq \mu(x)$ and $\mu(x) + t \leq 1)$ or $(t \leq \mu(y)$ and $\mu(y) + t \leq 1)$ .Thus $t \leq 0.5$,a contradiction.

(F2a) $\Rightarrow$ (F1a) : Let $(x+y)_{t \wedge r} \overline{\in} \mu$,then $\mu(x+y) < t \wedge r$.

Case I: If $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ then $\mu(x) \wedge \mu(y) < t \wedge r$

and consequently $\mu(x) < t$ or $\mu(y) < r$. It follows that $x_t \overline{\in} \mu$ or $y_r \overline{\in} \mu$. Thus $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$.

Case II: If $\mu(x+y) < \mu(x) \wedge \mu(y)$ then by (F2a) $0.5 \geq \mu(x) \wedge \mu(y)$. Taking $x_t \overline{\in} \mu$ or $y_r \overline{\in} \mu$ we get $t \leq \mu(x) \leq 0.5$ or $r \leq \mu(y) \leq 0.5$. It follows that $x_t \overline{q}\mu$ or $y_r \overline{q}\mu$ and thus $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$. This completes the proof.

(F1a') $\Leftrightarrow$ (F2a') is similar.

Now we prove (F1b) $\Leftrightarrow$ (F2b):

(F1b) $\Rightarrow$ (F2b): Let $(nx)_t \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ Let $\mu(nx) \vee 0.5 < t = \mu(x)$, then $0.5 < t \leq 1$. Thus $\mu(nx) < t$ or $(nx)_t \overline{\in} \mu$ implies $\mu(x) + t > 1$ or $x_t q\mu$, a contradiction Thus $\mu(nx) \vee 0.5 \geq \mu(x)$.

(F2b) $\Rightarrow$ (F1b): Let $(nx)_t \overline{\in} \mu$,then $\mu(nx) < t$

Case I: If $\mu(nx) \geq \mu(x)$ then $\mu(x) < t$ implies $x_t \overline{\in} \mu$

Case II: If $\mu(nx) < \mu(x)$ then as $\mu(nx) \vee 0.5 \geq \mu(x)$ so $\mu(x) \leq 0.5$. Now taking $x_t \overline{\in} \mu$ we get $\mu(x) \geq t$. Thus $t \leq \mu(x) \leq 0.5$ gives $x_t \overline{q} \mu$.

Consequently in each case $(nx)_t$ implies $x_t \overline{\in} \vee \overline{q} \mu$.

Theorem 3.4: Let A be a non empty subset of E. Then χ_A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$ fuzzy N-subgroup of E if and only if A is N-subgroup of E.

Proof : Let A be an N-subgroup of E. To prove χ_A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$ fuzzy N-subgroup of E.

Since A is an N-subgroup of E so we have χ_A is fuzzy N-subgroup of E. Hence for all x, y in A we have $\chi_A(x+y) \geq \chi_A(x) \wedge \chi_A$ and $\chi_A(-x) \geq \chi_A(x)$ and so $\chi_A(x+y) \vee 0.5 \geq \chi_A(x) \wedge \chi_A$ and $\chi_A(-x) \vee 0.5 \geq \chi_A(x)$. Again if $x \in A$ then for any $n \in N$ we have $nx \in A$ .which gives us $\chi_A(nx) \geq \chi_A(x)$ or $\chi_A(nx) \vee 0.5 \geq \chi_A(x)$. Also if $x \notin A$ then $\chi_A(x) = 0$ and then the result follows. Thus χ_A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$ fuzzy N-subgroup of E.

Conversely let χ_A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$ fuzzy N-subgroup of E. To show A is an N-subgroup of E. Since χ_A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$ fuzzy N-subgroup of E so we have for $x, y \in A, \chi_A(x+y) \vee 0.5 \geq \chi_A(x) \wedge \chi_A(y)$ .But $x, y \in A \Rightarrow \chi_A(x) = 1, \chi_A(y) = 1$ and so we have $\chi_A(x+y) = 1 \Rightarrow x + y \in A$. Again for $x \in A$ since $\chi_A(-x) \vee 0.5 \geq \chi_A(x)$ so similarly $-x \in A$.

Again for $n \in N, x \in A$ as $\chi_A(nx) \vee 0.5 \geq \chi_A(x)$ and $x \in A \Rightarrow \chi_A(x) = 1$ so $\chi_A(nx) = 1$ which implies $nx \in A$. Hence A is an N-subgroup of E.

Definition 3.5: Let μ be a fuzzy subset of an N – group E. Then μ is said to be an $(\overline{\in}\ \overline{\in} \vee \overline{q})$ fuzzy ideal of E if for all $n \in N, a, x, y \in E$ and $t, r \in (0,1]$

(F5a) $(x+y)_{t \wedge r} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$

(F5a’) $(-x)_t \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$

(F5b) $(x+y-x)_t \overline{\in} \mu$ implies $y_t \overline{\in} \vee \overline{q} \mu$

(F5c) $[n(a+x)-na]_t \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$

Theorem 3.6: : A fuzzy subset μ of an N-group E is an $(\overline{\in}\ \overline{\in} \vee \overline{q})$ fuzzy ideal of E if and only if for all $n \in N, a, x, y \in E$

(F6a) $\mu(x+y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$

(F6a’) $\mu(-x) \vee 0.5 \geq \mu(x)$

(F6b) $\mu(x+y-x) \vee 0.5 \geq \mu(y)$

(F6c) $\mu[n(x+a)-nx] \vee 0.5 \geq \mu(x)$

Proof : Here we will prove (F5b) $\Leftrightarrow$ (F6b). The others are similar.

(F5b) $\Rightarrow$ (F6b): Let $(x+y-x)_t \overline{\in} \mu$ implies $y_t \overline{\in} \vee \overline{q} \mu$

To show that $\mu(x+y-x) \vee 0.5 \geq \mu(y)$

Let $\mu(x+y-x) \vee 0.5 < t = \mu(y)$,then $0.5 < t \leq 1$ which implies $\mu(x+y-x) < t$

Or $(x+y-x)_t \overline{\in} \mu$ implies $\mu(y) + t > 1$ or $y_t q \mu$, a contradiction.

Thus $\mu(x+y-x)\vee 0.5 \geq \mu(y)$.

(F6b) $\Rightarrow$(F5b): Let $\mu(x+y-x)\vee 0.5 \geq \mu(y)$.

To show $(x+y-x)_t \overline{\in} \mu$ implies $y_t \overline{\in} \vee\overline{q}\mu$.

Let $(x+y-x)_t \overline{\in} \mu$ then we have $\mu(x+y-x)<t$.

Case I: If $\mu(x+y-x)\geq \mu(y)$ then $\mu(y)<t$ implies $y_t \overline{\in} \mu$.

Case II : If $\mu(x+y-x)<\mu(y)$ then $\mu(y)\leq 0.5$. Now taking $y_t \in \mu$ we get $\mu(y)\geq t$. Thus $t\leq \mu(y)\leq 0.5$ gives $y_t\overline{q}\mu$. Consequently in each case $(x+y-x)_t \overline{\in} \mu$ implies $y_t \overline{\in} \vee\overline{q}\mu$.

Theorem 3.7 : Let A be a non empty subset of N-group E. Then A is an ideal of E if and only if χ_A is an $(\overline{\in}, \overline{\in} \vee\overline{q})$ fuzzy ideal of E .

Proof : Let A be an ideal of E. Hence χ_A is a fuzzy normal subgroup of E. So we have

$$\chi_A(x+y)\geq \chi_A(x)\wedge\chi_A(y);$$

$$\chi_A(-x)\geq \chi_A(x) \text{ and } \chi_A(x+y-x)\geq \chi_A(y)$$

Consequently,

$$\chi_A(x+y)\vee 0.5\geq \chi_A(x)\wedge\chi_A(y);$$

$$\chi_A(-x)\vee 0.5\geq \chi_A(x) \text{ and}$$

$$\chi_A(x+y-x)\vee 0.5\geq \chi_A(y)$$

Again for $n\in N, a, x\in E$ if $x\in A$ then as A is an ideal of E so $n(a+x)-na\in A$ implies $\chi_A(n(a+x)-na)=\chi_A(x)=1$. So we have $\chi_A(n(a+x)-na)\vee 0.5\geq \chi_A(x)$

Again if $x\notin A$ then $\chi_A(x)=0$ and so $\chi_A(n(a+x)-na)\geq \chi_A(x)$.and so $\chi_A(n(a+x)-na)\vee 0.5\geq \chi_A(x)$.

Conversely , if χ_A is an $(\overline{\in}, \overline{\in} \vee\overline{q})$ fuzzy ideal of E then χ_A is a fuzzy normal subgroup of E and so we have A is normal subgroup of E. Again for $n\in N, x\in A, a\in E$ we have

$$\chi_A(n(a+x)-na)\vee 0.5\geq \chi_A(x)=1$$

which gives $\chi_A(n(a+x)-na)=1$ implies $n(a+x)-na\in A.$ Hence A is an ideal of E.

References:

[1] L.A.Zadeh, *Fuzzy Sets, Information and Control*,8(1965),338-353

[2] A.Rosenfield, *Fuzzy groups, J.Math,Anal*. Appl35(1971)512-517

[3] Wang-Jin Liu, *Fuzzy invariant Subgroups and Fuzzy ideals, Fuzzy sets and Systems*, 8(1982),133-139

[4] S. Abou Zaid, On *Fuzzy sub nearing and ideals*, FSS,44(1991),139-146

[5] Helen K. Saikia and Lila K. Barthakur, *On Fuzzy N-subgroups and Fuzzy ideals of near-rings and near-ring groups*, Journal of Fuzzy Mathematics Vol11, No3(2003)567-580

[6] P.P.Ming and L.Y. Ming:Fuzzy topology I,*Neighbourhood structure of a fuzzy point and Moore-Smith Convergence*, J. Math. Anal. Appl,76(1980)571-599

[7] S.K. Bhakat, P. Das, *On the definition of a fuzzy subgroup*. FSS,51(1992)235-241

[8] S.K. Bhakat, P.Das, *Fuzzy Subgroups and ideals redefined*:FSS,81(1996)383-393

[9] Zhan Jian-ming and Davvaz B,*Generalized fuzzy ideals of near-rings*, Appl. Math, J. Chinea Univ.2009,24(3):343-349

[10] Luo Fen and Zhan Jian-ming,*Redefined generalized fuzzy R-Subgroups of near-rings*, Italian Journal of pure and Applied Mathematics-N,30-2013(33-42)