Mathematical Analysis of Multi-phase single server and Multi-phase multi server: Comparison study

Nityangini Jhala[1], Pravin Bhathawala[2]

Assistant Professor, Department of Applied Sciences and Humanities, Parul University, Waghodia, Vadodara, Gujarat, India[1]

Retd. Professor / Head, Department of Mathematics, VNSGU, Surat, Gujarat, India[2]

Abstract: *Queues (waiting lines) are a part of everyday life. The Queuing models are very helpful for determining how to operate a queuing system in the most effective way if the waiting time of customers is much. In this research paper, we will prove that multi-phase multi-server queuing model is better as compared to multi-phase single server queuing model.*

Keywords: *$M/E_k/1$ model, $M/E_k/s$ model, multi phase single server and multi phase multi server models*

I. INTRODUCTION

Delays and queuing problems are most common features not only in our daily-life situations such as at a bank or supermarket, at a hospital, in public transportation or in a traffic jam but also in more technical environments, such as in manufacturing, computer networking and telecommunications.

Queuing theory is the mathematical study of waiting lines, or queues. A queuing model is constructed so that queue lengths and waiting time can be predicted. Queuing theory is generally considered a branch of operations research because the results are often used when making business decisions about the resources needed to provide a service. Queuing theory examines every component of waiting in line to be served, including the arrival process, service process, number of servers, number of system places and the number of "customers" (which might be people, data packets, cars, etc.) Queuing theory attempts to solve problems based on a scientific understanding of the problems and solving them in optimal manner so that facilities are fully utilized and waiting time is reduced to minimum possible. Waiting time (or queuing) theory models can recommend arrival of customers to be serviced, setting up of workstations, requirement of manpower etc. based on probability theory.

II. RELATED WORK

Emmanuel Ekpenyong and Nse Udoh[1] extended the formulas of multi-phase single server queuing model to multi-phase multi server queuing model. The extension results in a new Queuing System of MultiServer with Multiple Phases under the conditions of First Come First Served, infinite population source, Poisson arrivals and Erlang service time.

S. Vijay Prasad and V.H. Badshah[2] derived the total cost with assumption of certain Waiting cost and proved that the expected total cost is less for single queue multi server model as comparing with multi queue multi server model. Mathematically it has been proved by the examples.

Priyangika J.S.K.C and Cooray T.M.J.A[3] explained the analysis of Queuing systems for the empirical data of supermarket checkout service unit using queuing theory. The main purpose of this paper is to review the application of queuing theory and to evaluate the parameters involved in the service unit for the sales checkout operation in the Supermarket. Therefore, a mathematical model is developed to analyze the performance of the checking out service unit.

Saima Mustafa and S.un Nisa[4] analyzed using queuing analysis and queuing simulation in three departments. Single server and multiple server queuing models have been used for these analysis. Analysis shown that waiting time of patients in queue is greater in pharmacy department in both days as compared to other departments. From the analysis it has also been observed that waiting time of patients could be reduced by using multiple servers rather than single server queuing model.

Dr.K.L.Muruganantha Prasad and B.Usha[5] calculated average queuelength, average number of customer in the system, average customer waiting time and average number of customer time spent in the queue in kanyakumari district at various places. Comparing these two models the values of M/M/1 model is greater than the values of M/D/1 model.

III. PROPOSED METHODOLOGY AND DISCUSSION

We will make the following assumptions for queuing system in accordance with queuing theory.

1. Arrivals follow a Poisson probability distribution at an average rate of λ customers per unit of time.

2. The queue discipline is First-Come, First-Served (FCFS) basis by any of the servers. There is no priority classification for any arrival.

3. Service times are distributed exponentially, with an average of μ customers per unit of time.

4. There is no limit to the number of the queue (infinite).

5. The service providers are working at their full capacity.

6. The average arrival rate is greater than average service rate.

7. Service rate is independent of line length; service providers do not go faster because the line is longer.

$M/E_K/1$ queuing model:(multi-phase single-server)

λ: The mean customers arrival rate

μ: The mean service rate

The average number of customers in the queue:

$$L_{q1} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right)$$

The average number of customers in the system:

$$L_{s1} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu}$$

The average waiting time in the queue:

$$W_{q1} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right)$$

The average time spent in the system, including the waiting time in the queue:

$$W_{s1} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right) + \frac{1}{\mu}$$

$M/E_K/s$ queuing model:(multi-phase multi-server)

λ: The mean customers arrival rate

μ: The mean service rate

The average number of customers in the queue:

$$L_{qs} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right)$$

The average number of customers in the system:

$$L_{ss} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right) + \frac{\lambda}{s\mu}$$

The average waiting time in the queue:

$$W_{qs} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{s\mu(s\mu-\lambda)}\right)$$

The average time spent in the system, including the waiting time in the queue:

$$W_{ss} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{s\mu(s\mu-\lambda)}\right) + \frac{1}{s\mu}$$

(i) We shall show that the number of customers waiting in the queue in multi-phase multi-server is less than multi-phase single-server i.e., $L_{qs} \leq L_{q1}$

i.e., $\left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right) \leq \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right)$ …(1)

We will prove the result by induction.

Let us prove the case for s=1

Equation (1) becomes $0 \leq 0$

So, eq(1) works for $s = 1$

Let us assume that the result holds for $s = p$

i.e., $\left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{p\mu(p\mu-\lambda)}\right) \leq \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right)$

We want to show that the result is true for $s = p + 1$

i.e., $\left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{[(p+1)\mu][(p+1)\mu-\lambda]}\right) \leq \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right)$

Now, $p+1 > p$

$(p+1)\mu > p\mu$

$(p+1)\mu - \lambda > p\mu - \lambda$

$(p+1)[(p+1)\mu - \lambda] > (p+1)(p\mu - \lambda)$

$> p(p\mu - \lambda)$

So, $(p+1)[(p+1)\mu - \lambda] > p(p\mu - \lambda)$

$$\frac{1}{(p+1)[(p+1)\mu - \lambda]} < \frac{1}{p(p\mu - \lambda)}$$

$$\left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu}\left[\frac{1}{(p+1)[(p+1)\mu - \lambda]}\right] < \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu}\left[\frac{1}{p(p\mu - \lambda)}\right]$$

which proves the result for $s = p+1$

So, by mathematical induction,

$\left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right) \leq \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right)$

i.e., $L_{qs} \leq L_{q1}$

(ii) We shall show that the number of customers waiting in the system in multi-phase multi-server is less than multi-phase single-server i.e., $L_{ss} \leq L_{s1}$

i.e., $\left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right) + \frac{\lambda}{s\mu} \leq \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu}$…(2)

We will prove the result by induction.

Let us prove the case for s=1

Equation (2) becomes $0 \leq 0$

So, eq(2) works for $s = 1$

Let us assume that the result holds for $s = p$

i.e., $\left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{p\mu(p\mu-\lambda)}\right) + \frac{\lambda}{p\mu} \leq \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu}$ ... (3)

We want to show that the result is true for $s = p+1$

i.e., $\left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{[(p+1)\mu][(p+1)\mu-\lambda]}\right) + \frac{\lambda}{(p+1)\mu} \leq \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu}$

Now, $p+1 > p$

$(p+1)\mu > p\mu$

$(p+1)\mu - \lambda > p\mu - \lambda$

$(p+1)[(p+1)\mu - \lambda] > (p+1)(p\mu - \lambda)$

$> p(p\mu - \lambda)$

So, $(p+1)[(p+1)\mu - \lambda] > p(p\mu - \lambda)$

$$\frac{1}{(p+1)[(p+1)\mu - \lambda]} < \frac{1}{p(p\mu - \lambda)}$$

$\left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu}\frac{1}{(p+1)[(p+1)\mu-\lambda]} < \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu}\frac{1}{p(p\mu-\lambda)}$ …(4)

Also, $p+1 > p$

$$\frac{1}{p+1} < \frac{1}{p}$$

$\frac{\lambda}{\mu}\left[\frac{1}{p+1}\right] < \frac{\lambda}{\mu}\left[\frac{1}{p}\right]$ …(5)

Adding (4) and (5), $\left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu}\frac{1}{(p+1)[(p+1)\mu-\lambda]} + \frac{\lambda}{\mu}\left[\frac{1}{p+1}\right] < \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu}\frac{1}{p(p\mu-\lambda)} + \frac{\lambda}{\mu}\left[\frac{1}{p}\right]$ …(6)

Using (3) and (6), the result is proved for $s = p+1$

Similarly, we can prove by induction that

(iii) The waiting time of customers in the queue in case of multiple server is less as compared to single server i.e., $W_{qs} \leq W_{q1}$ and

(iv)) The waiting time of customers in the system in case of multiple server is less as compared to single server $W_{ss} \leq W_{s1}$

IV. RESULTS AND CONCLUSION:

In this paper, we have shown that multi-server multi-phase model is better than multi-phase single-server model. The waiting time of the customers is less in the former as compared to the latter. Also, the number of customers waiting in the queue are also less in multi-phase multi-server model.

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