Graceful Labeling of Quarilateral Snakes

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Abstract: *The graceful labeling of a graph G with q edges means that there is an injection g : V(G) to { 0,1, 2, ..., q} such that, when each edge uv is assigned the label |g(u)-g(v)|, the resulting edge labels are {1, 2, 3, ...q}. A graph which admits an graceful labeling is called an graceful graph. In this paper we will prove alternate quadrilateral snakes* $A(QS_n)$*, double quadrilateral snakes* $D(QS_n)$*, double alternate quadrilateral snakes* $DA(QS_n)$ *are graceful.*

Keywords: *alternate quadrilateral snakes, double quadrilateral snakes, double alternate quadrilateral snakes, graceful labeling.*

AMS Mathematics Subject Classification (2010): 05C78

1 INTRODUCTION

One of the most interesting problem in graph theory deals with labeling the vertices of a graph subject to certain constraints. A **vertex labeling** of a graph **G=(V,E)** is an assignment f of labels to the vertices of G that induces for each edge $uv \in E(G)$ a label depending on the vertex labels $f(u)$ and $f(v)$. Graceful labeling was introduced by Rosa [7] in 1967. A graceful labeling of a graph G with n-edges is an injection $g : V(G) \rightarrow \{0, 1, 2, ..., n\}$ with the property that the resulting edge labels are also distinct where an edge incident with vertices u and v is assigned the label $| g(u)-g(v) |$. A graph which admits graceful labeling is called a graceful graph.

A triangular cactus is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block-cut-vertex graph is a path. By a block-cut-vertex graph of a graph G we mean the graph whose vertices are the blocks and cut-vertices of G where two vertices are adjacent if and only if one vertex is a block and the other is a cut-vertex belonging to the block. In 2001 Barrientos [1] introduced kC_4 -snake graph as a generalization of the concept of triangular snake introduced by Rosa [8] and he proves that kC_4 -snakes are graceful. A kC_4 -snake is a connected graph with k blocks, each of the block is isomorphic to the cycle C_n, such that the block-cut-vertex graph is a path. Let $u_1, u_2, u_3, ..., u_{k-1}$ be the consecutive cut-vertices of G and d_i be the distance between u_i and u_{i+1} in G for $1\leq i \leq k - 2$ the string $(d_1,d_2, . . . d_{k-2})$ of integers characterizes the graph G in the class of n-cyclic snakes. If the string of given kC_n -snake is $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \ldots, \lfloor \frac{n}{2} \rfloor)$, we say that kC_n -snake is linear. A quadrialteral snake is a kC_4 -snake graph with string (1, 1, 1, ..., 1) and Gnanajothi [4] had shown quadrilateral snakes are graceful. In this paper, we will prove alternate quadrilateral snakes $A(QS_n)$, double quadrilateral snakes $D(QS_n)$ and double alternate quadrilateral snakes $DA(QS_n)$ are graceful.

Definition 1.1

A quadrilateral snake QS_n is obtained from a path $u_1, u_2, u_3, ..., u_n$ by joining u_i, u_{i+1} to new vertices v_i and w_i respectively and adding edges $v_i w_i$ for $i = 1, 2, ..., n-1$. That is every edge of a path is replaced by a cycle C_4.

Definition 1.2

An alternate quadrilateral snake $A(QS_n)$ is obtained from a path $u_1, u_2, u_3, ..., u_n$ by joining u_i, u_{i+1} (alternatively) to new vertices v_i and w_i respectively and adding edges $v_i w_i$, where $1\leq i \leq n-1$ for even n and $1\leq i \leq n-2$ for odd n. That is every alternate edge of a path is replaced by a cycle C_4.

Definition 1.3

A double quadrilateral snake $D(QS_k)$ is obtained from two quadrilateral snakes that have a common path.

Definition 1.4

An alternate double quadrilateral snake $DA(QS_k)$ is obtained from two alternative quadrilateral snakes that have a common path.

2 MAIN RESULTS

Theorem 2.1.

Alternate quadrilateral snakes $A(QS_n)$ is graceful.

Proof:

Let $A(QS_n)$ be an alternate quadrilateral snake obtained from a path $u_1, u_2, u_3, ..., u_n$ by joining u_i, u_{i+1} (alternatively) to new vertices v_i and w_i respectively and adding edges $v_i w_i$ where $1\leq i \leq n-1$

for even n and $1\le i \le n\text{-}2$ for odd n. Then, |E(G)|=

$$= \begin{cases} \frac{5n-2}{2}, & n \equiv 0(mod2) \\ \frac{5n-5}{2}, & n \equiv 1(mod2) \end{cases}$$

Case (i) : $n \equiv 0$ (mod2)

Define g : V(G) → $\{0, 1, 2, \ldots, \frac{5n}{2} - 1\}$ by

$g(u_{2i-1}) = 2(i-1) \quad 1\le i \le \frac{n}{2}$

$g(u_{2i}) = \frac{5n}{2} - 3i \quad 1\le i \le \frac{n}{2}$

$g(v_{2i-1}) = \frac{5n}{2} - 3i + 2 \quad 1\le i \le \frac{n}{2}$

$g(w_{2i-1}) = 2i - 1 \quad 1\le i \le \frac{n}{2}$

Let S={ $g(u_{2i-1})$ | $1\le i \le n/2$, $g(u_{2i})$ | $1\le i \le n/2$, $g(v_{2i-1})$ | $1\le i \le n/2$, $g(w_{2i-1})$ | $1\le i \le n/2$}

={ $g(u_1), g(u_3), \ldots, g(u_{n-1}), g(u_2), g(u_4), \ldots, g(u_n), g(v_1), g(v_3), \ldots, g(v_{n-1}), g(w_1), g(w_3), \ldots, g(w_{n-1})$}

$=\{0, 2, 4, \ldots, n\text{-}2, \frac{5n}{2} - 3, \frac{5n}{2} - 6, \ldots, n, \frac{5n}{2} - 1, \frac{5n}{2} - 4, \ldots, n+2, 1, 3, 5, \ldots, n\text{-}1\}$

$=\{0, 1, 2, \ldots, n\text{-}2, n\text{-}1, n, \ldots, \frac{5n}{2} - 1\}$

(1)

To prove edge labels are distinct and even

The vertices u_{2i-1} and u_{2i}, $1\le i \le n/2$ induces the edge labels $\{\frac{5n}{2} - 5i + 2, 1\le i \le n/2\} = \{\frac{5n}{2} - 3, \frac{5n}{2} - 8, \ldots, 2\}$

The vertices u_{2i} and u_{2i+1}, $1\le i \le (n/2)\text{-}1$ induces the edge labels $\{\frac{5n}{2} - 5i, 1\le i \le (n/2)\text{-}1\} = \{\frac{5n}{2} - 5, \frac{5n}{2} - 10, \ldots, 5\}$

The vertices u_{2i-1} and v_{2i-1}, $1\le i \le n/2$ induces the edge labels $\{\frac{5n}{2} - 5i + 4, 1\le i \le n/2\} = \{\frac{5n}{2} - 1, \frac{5n}{2} - 6, \ldots, 4\}$

The vertices w_{2i-1} and u_{2i}, $1\le i \le n/2$ induces the edge labels $\{\frac{5n}{2} - 5i + 1, 1\le i \le n/2\} = \{\frac{5n}{2} - 4, \frac{5n}{2} - 9, \ldots, 1\}$

The vertices v_{2i-1} and w_{2i-1}, $1\le i \le n/2$ induces the edge labels $\{\frac{5n}{2} - 5i + 3, 1\le i \le n/2\} = \{\frac{5n}{2} - 2, \frac{5n}{2} - 7, \ldots, 3\}$

So we obtain all the edge labels $\{1, 2, \ldots, \frac{5n}{2} - 2, \frac{5n}{2} - 1\}$ (2)

Therefore, from (1) and (2) it is clear that Alternate quadrilateral snakes A(QS_n) is graceful for even n.

Case (ii) $n \equiv 1$ (mod2)

Define g : V(G) → $\{0, 1, 2, \ldots, \frac{5(n-1)}{2}\}$ by

$g(u_{2i-1}) = 2(i-1) \quad 1\le i \le \lceil \frac{n}{2} \rceil$

$g(u_{2i}) = \frac{5(n-1)}{2} - 3i + 1 \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

$g(v_{2i-1}) = \frac{5(n-1)}{2} - 3i + 3 \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

$g(w_{2i-1}) = 2i - 1 \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

Proof same as previous theorem. Therefore, Alternate quadrilateral snakes A(QS_n) is graceful for odd n.

Hence, Alternate quadrilateral snakes A(QS_n) is graceful.

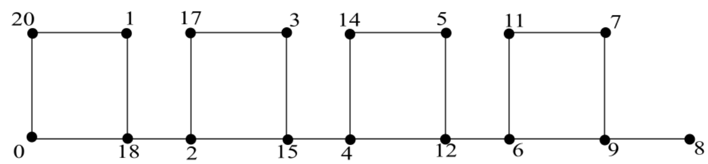

Figure 1 : Graceful labeling of A(QS_9)

Theorem 2.2.

Double quadrilateral snakes D(QS_n) is graceful.

Proof*:*

Let D(QS_n) be an double quadrilateral snake obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining u_i and u_{i+1} to new vertices v_i, x_i and w_i, y_i respectively and adding edges $v_i w_i$ and $x_i y_i$, where $1\le i \le n\text{-}1$. Then, |E(G)|= 7(n-1)

Define g : V(G) → {0, 1, 2, ..., 7(n-1)} by

$g(u_{2i-1}) = 7(i\text{-}1) \quad 1\le i \le \lceil \frac{n}{2} \rceil$

$g(u_{2i}) = 7(n\text{-}1)\text{-}3\text{-}7(i\text{-}1) \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

$g(w_{2i-1}) = 7(i\text{-}1)+1 \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

$g(w_{2i}) = 7(n\text{-}1)\text{-}4\text{-}7(i\text{-}1) \quad 1\le i \le \lceil \frac{n}{2} \rceil\text{-}1$

$g(v_{2i-1}) = 7(n\text{-}1)\text{-}7(i\text{-}1) \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

$g(v_{2i}) = 7(i\text{-}1)+4 \quad 1\le i \le \lceil \frac{n}{2} \rceil\text{-}1$

$g(x_{2i-1}) = 7(n\text{-}1)\text{-}7(i\text{-}1)\text{-}2 \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

$g(x_{2i}) = 7(i\text{-}1)+6 \quad 1\le i \le \lceil \frac{n}{2} \rceil\text{-}1$

$g(y_{2i-1}) = 7(i\text{-}1)+3 \quad 1\le i \le \lfloor \frac{n}{2} \rfloor$

$g(y_{2i}) = 7(n\text{-}1)\text{-}7(i\text{-}1)\text{-}6 \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil\text{-}1$

Case(i) : n is even

Let S={ $g(u_{2i-1})$ | $1\le i \le n/2$, $g(u_{2i})$ | $1\le i \le n/2$, $g(w_{2i-1})$ | $1\le i \le n/2$, $g(v_{2i-1})$ | $1\le i \le n/2$, $g(x_{2i-1})$ | $1\le i \le n/2$, $g(y_{2i-1})$ | $1\le i \le n/2$}

={ $g(u_1)$, $g(u_3)$, ..., $g(u_{n-1})$, $g(u_2)$, $g(u_4)$, ..., $g(u_n)$, $g(w_1)$, $g(w_3)$, ..., $g(w_{n-1})$, $g(w_2)$, $g(w_4)$, ..., $g(w_{n-2})$, $g(v_1)$, $g(v_3)$, ..., $g(v_{n-1})$, $g(v_2)$, $g(v_4)$, ..., $g(v_{n-2})$, $g(x_1)$, $g(x_3)$, ..., $g(x_{n-1})$, $g(x_2)$, $g(x_4)$, ..., $g(x_{n-2})$, $g(y_1)$, $g(y_3)$, ..., $g(y_{n-1})$, $g(y_2)$, $g(y_4)$, ..., $g(y_{n-2})$}

{0, 7, 14, ..., $\frac{7}{2}n$-7, 7n-10, 7n-17, ..., $\frac{7}{2}n-3$, 1, 8, ..., $\frac{7}{2}n$ -6, 7n-11, 7n-18, ..., $\frac{7}{2}n$ +3, 7n-7, 7n-14, ..., $\frac{7}{2}n$, 4, 11, ..., $\frac{7}{2}n$ -10, 7n-9, 7n-16, ..., $\frac{7}{2}n$ -2, 6, 13, ..., $\frac{7}{2}n$ -8, 3, 10, ..., $\frac{7}{2}n$ -4, 7n-13, 7n-20, ..., $\frac{7}{2}n$ +1}={0, 1, 3, 4, ..., $\frac{7}{2}n$ -3, $\frac{7}{2}n$ -2, ..., 7n-7} (3)

To prove edge labels are distinct

The vertices u_{2i-1} and u_{2i}, $1\le i \le n/2$ induces the edge labels {7n-14i+4 $1\le i \le n/2$}= {7n-10, 7n-24, ..., 4}

The vertices u_{2i} and u_{2i+1}, $1\le i \le (n/2)\text{-}1$ induces the edge labels {7n-14i-3, $1\le i \le (n/2)\text{-}1$}={7$n$-17, 7$n$-31, ..., 11}

The vertices u_{2i-1} and v_{2i-1}, $1\le i \le n/2$ induces the edge labels {7n-14i+7, $1\le i \le n/2$}= {7n-7, 7n-21, ..., 7}

The vertices u_{2i-1} and x_{2i-1}, $1\le i \le n/2$ induces the edge labels {7n-14i+5, $1\le i \le n/2$}= {7n-9, 7n-13, ..., 5}

The vertices u_{2i+1} and w_{2i}, $1\le i \le (n/2)\text{-}1$ induces the edge labels {7n-14i-4, $1\le i \le (n/2)\text{-}1$}= {7$n$-18, 7$n$-32, ..., 10}

The vertices u_{2i+1} and y_{2i}, $1\le i \le (n/2)\text{-}1$ induces the edge labels {7n-14i-6, $1\le i \le (n/2)\text{-}1$}= {7$n$-20, 7$n$-34, ..., 8}

The vertices u_{2i} and w_{2i-1}, $1\le i \le n/2$ induces the edge labels {7n-14i +3, $1\le i \le n/2$}= {7n-11, 7n-25, ..., 3}

The vertices u_{2i} and y_{2i-1}, $1\le i \le n/2$ induces the edge labels {7n-14i +1, $1\le i \le n/2$}= {7n-13, 7n-27, ..., 1}

The vertices u_{2i} and v_{2i}, $1\le i \le (n/2)\text{-}1$ induces the edge labels {7n-14i, $1\le i \le (n/2)\text{-}1$}= {7$n$-14, 7$n$-28, ..., 14}

The vertices u_{2i} and x_{2i}, $1\le i \le (n/2)\text{-}1$ induces the edge labels {7n-14i-21, $1\le i \le (n/2)\text{-}1$}= {7$n$-16, 7$n$-30, ..., 12}

The vertices v_{2i-1} and w_{2i-1}, $1\le i \le n/2$ induces the edge labels {7n-14i +6, $1\le i \le n/2$}= {7n-8, 7n-22, ..., 6}

The vertices v_{2i} and w_{2i}, $1\le i \le (n/2)\text{-}1$ induces the edge labels {4n-8i+6, $1\le i \le (n/2)\text{-}1$}= {7$n$-15, 7$n$-29, ..., 13}

The vertices x_{2i-1} and y_{2i-1}, $1\le i \le n/2$ induces the edge labels {7n-14i+2, $1\le i \le n/2$}= {7n-12, 7n-26, ..., 2}

The vertices x_{2i} and y_{2i}, $1\le i \le (n/2)\text{-}1$ induces the edge labels {7n-14i-5, $1\le i \le (n/2)\text{-}1$}= {7$n$-19, 7$n$-33, ..., 9}

So we obtain all the edge labels {1, 2, 3, …, 7n-6, 7n-7} (4)

Hence, from (3) and (4) it is clear that double quadrilateral snakes D(QS$_n$) is graceful for even n.

Case(ii): n is odd

The proof is similar to the proof in case(i) as the change is only in the range of values.

Hence, Double quadrilateral snakes D(QS$_n$) is graceful.

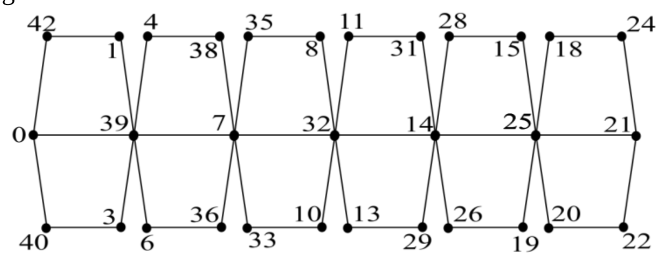

Figure 2 : Graceful labeling of D(QS$_7$)

Theorem 2.3.

Double alternate quadrilateral snakes DA(QS$_n$) is graceful.

Proof:

Let DA(QS$_n$) be an double alternate quadrilateral snake obtained from a path u_1, u_2, u_3, ..., u_n by joining u_i and u_{i+1} (alternatively) to new vertices v_i x_i and w_i y_i respectively and adding edges v_i w_i and x_i y_i where $1\le i \le n\text{-}1$ for even n and $1\le i \le n\text{-}2$ for odd n. Then, |E(G)|= $\begin{cases} 4n-1, & n \equiv 0(mod 2) \\ 4n-4, & n \equiv 1(mod 2) \end{cases}$

Case (i) : $n \equiv 0$ (mod2)

Define g : V(G) ⟶ {0, 1, 2, …, 4n-1} by

$g(u_{2i-1}) = 4(i\text{-}1) \quad 1\le i \le n/2$

$g(u_{2i}) = 4n\text{-}4i \quad 1\le i \le n/2$

$g(w_{2i-1}) = 4i - 3 \quad 1\le i \le n/2$

$g(v_{2i-1}) = 4n\text{-}4i\text{+}3 \quad 1\le i \le n/2$

$g(x_{2i-1}) = 4n\text{-}4i\text{+}1 \quad 1\le i \le n/2$

$g(y_{2i-1}) = 4i\text{-}1 \quad 1\le i \le n/2$

Let S={ $g(u_{2i-1})$ | $1\le i \le n/2$, $g(u_{2i})$ | $1\le i \le n/2$, $g(w_{2i-1})$ | $1\le i \le n/2$, $g(v_{2i-1})$ | $1\le i \le n/2$, $g(x_{2i-1})$ | $1\le i \le n/2$, $g(y_{2i-1})$ | $1\le i \le n/2$}

={ $g(u_1)$, $g(u_3)$, ..., $g(u_{n-1})$, $g(u_2)$, $g(u_4)$, ..., $g(u_n)$, $g(w_1)$, $g(w_3)$, ..., $g(w_{n-1})$, $g(v_1)$, $g(v_3)$, ..., $g(v_{n-1})$, $g(x_1)$, $g(x_3)$, ..., $g(x_{n-1})$, $g(y_1)$, $g(y_3)$, ..., $g(y_{n-1})$ }

={0, 4, ..., 2n-4, 4n-4, 4n-8, ..., 2n, 1, 5, ..., 2n-3, 4n-1, 4n-5, ..., 2n+3,4n-3, 4n-7, ..., 2n+1, 3, 7, ..., 2n-1}

={0, 1, 3, 4, 5, 7, ..., 2n-1, 2n, 2n+1, ..., 4n-3, 4n-1} (5)

To prove edge labels are distinct

The vertices u_{2i-1} and u_{2i}, $1\leq i \leq n/2$ induces the edge labels $\{4n-8i+4$ $1\leq i \leq n/2\}$= {4n-4, 4n-12, ..., 4}

The vertices u_{2i} and u_{2i+1}, $1\leq i \leq (n/2)-1$ induces the edge labels {4n-8i, $1\leq i \leq (n/2)-1$}={4n-8, 4n-16, ..., 8}

The vertices u_{2i-1} and v_{2i-1}, $1\leq i \leq n/2$ induces the edge labels {4n-8i+7, $1\leq i \leq n/2$}= {4n-1, 4n-9, ..., 7}

The vertices u_{2i-1} and x_{2i-1}, $1\leq i \leq n/2$ induces the edge labels {4n-8i+5, $1\leq i \leq n/2$}= {4n-3, 4n-11, ..., 5}

The vertices u_{2i} and w_{2i-1}, $1\leq i \leq n/2$ induces the edge labels {4n-8i+3, $1\leq i \leq n/2$}= {4n-5, 4n-13, .., 3}

The vertices u_{2i} and y_{2i-1}, $1\leq i \leq n/2$ induces the edge labels {4n-8i+1, $1\leq i \leq n/2$}= {4n-7, 4n-15, .., 1}

The vertices v_{2i-1} and w_{2i-1}, $1\leq i \leq n/2$ induces the edge labels {4n-8i+6, $1\leq i \leq n/2$}= {4n-2, 4n-10, .., 6}

The vertices x_{2i-1} and y_{2i-1}, $1\leq i \leq n/2$ induces the edge labels {4n-8i+2, $1\leq i \leq n/2$}= {4n-6, 4n-14, .., 2}

So we obtain all the edge labels {1, 2, 3, …, 4n-2, 4n-1} (6)

Hence, from (5) and (6) it is clear that double alternate quadrilateral snakes DA(QS_n) is graceful for even n.

Case (ii): $n \equiv 1 \pmod 2$

Define g : V(G) → {0, 1, 2, …, 4n-4} by

$g(u_{2i-1}) = 4(i-1) \quad 1\leq i \leq \left\lceil \frac{n}{2} \right\rceil$

$g(u_{2i}) = 4n-4i-3 \quad 1\leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$g(w_{2i-1}) = 4i-3 \quad 1\leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$g(v_{2i-1}) = 4n-4i \quad 1\leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$g(x_{2i-1}) = 4n-4i-2 \quad 1\leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$g(y_{2i-1}) = 4i-1 \quad 1\leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

Proof same as previous theorem. Therefore, Double alternate quadrilateral snakes DA(QS_n) is graceful for odd n.

Hence, Double alternate quadrilateral snakes DA(QS_n) is graceful.

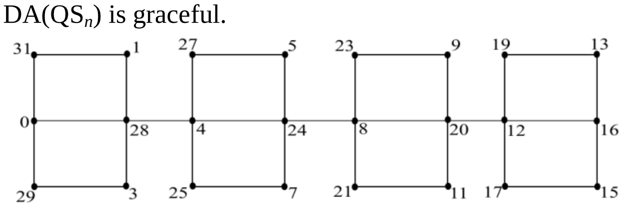

Figure 3 : Graceful labeling of DA(QS_8)

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