Lie's Symmetries of (2+1)dim PDE

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Abstract: In this paper we apply Lie's Classical Method to a (2+1) dimensional PDE equation to identify all possible solution for which this equation admits an exact solution and we obtain Lie algebra of infinitesimal symmetries is spanned by the six vector fields. We conclude that there is an infinite group of point transformations are invariant using invariant form method.We obtained local symmetry classifications.
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1. Introduction

In recent years there have been so many research to find symmetries for system of nonlinear (2+1) dimensional PDEs. In this paper we consider the (2+1) dimensional PDE

$$\begin{aligned} w_y - v &= 0, \\ w_x - u &= 0, \\ w_t + \frac{1}{2}u^2 + v_y &= 0. \end{aligned} \tag{1}$$

The above system (1) is equivalent to

$$w_t + \frac{1}{2}w_x^2 + w_{yy} = 0. \tag{2}$$

In the present work we first apply Lie classical method (Bluman and Cole[1], Bluman and Kumei[2], George M.Murphy [3], Olver[4], Bluman and Anco[5], Olver P.J[6,8] , Ibragimov[7], Daniel Zwillinger[9], Sachdev.P. L.[10,11], and Tesdall M. and Hunter.K[12]) to the second order PDE(2) and show that this system is invariant under a group containing six parameters using invariant form method.

1

We consider a scalar second order PDE in three variables

$$F(x_1, x_2, x_3, u, u_1, u_2, u_3, u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0. \tag{3}$$

Here $u_1 = \frac{\partial u}{\partial x_1}$, $u_2 = \frac{\partial u}{\partial x_2}$, $u_3 = \frac{\partial u}{\partial x_3}$, $u_{11} = \frac{\partial^2 u}{\partial x_1^2}$, $u_{12} = \frac{\partial^2 u}{\partial x_1 \partial x_2}$, $u_{13} = \frac{\partial^2 u}{\partial x_1 \partial x_3}$, $u_{22} = \frac{\partial^2 u}{\partial x_2^2}$, $u_{23} = \frac{\partial^2 u}{\partial x_2 \partial x_3}$, $u_{33} = \frac{\partial^2 u}{\partial x_3^2}$. The one-parameter Lie group of transformations

$$\begin{aligned}
x_1^* &= x_1 + \epsilon\xi_1(x_1, x_2, x_3, u,) + O(\epsilon^2),\\
x_2^* &= x_2 + \epsilon\xi_2(x_1, x_2, x_3, u,) + O(\epsilon^2),\\
x_3^* &= x_3 + \epsilon\xi_3(x_1, x_2, x_3, u,) + O(\epsilon^2),\\
u^* &= u + \epsilon\eta(x_1, x_2, x_3, u) + O(\epsilon^2),
\end{aligned} \tag{4}$$

leaves PDE (2) invariant if and only if its second extension also leaves (2) invariant.

Let

$$X = \sum_{i=1}^{3} \xi_i(x_1, x_2, x_3, u)\frac{\partial}{\partial x_i} + \eta(x_1, x_2, x_3, u)\frac{\partial}{\partial u} \tag{5}$$

be the infinitesimal generator of (4). Let

$$\begin{aligned}
X^{(2)} = &\sum_{i=1}^{3} \xi_i(x_1, x_2, x_3, u)\frac{\partial}{\partial x_i} + \eta(x_1, x_2, x_3, u)\frac{\partial}{\partial u} + \sum_{i=1}^{3} \eta_i^{(1)}(x_1, x_2, x_3, u, u_1, u_2, u_3)\frac{\partial}{\partial u_i}\\
&+ \cdots + \sum_{i=1}^{3} \eta_{i_1 i_2}^{(2)}(x_1, x_2, x_3, u, u_1, u_2, u_3, u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33})\frac{\partial}{\partial u_{i_1 i_2}}
\end{aligned} \tag{6}$$

be the second extended infinitesimal generator of (5)where $\eta_i^{(1)}$ is given by

$$\begin{aligned}
\eta_1^{(1)} &= U_{x_1} + U_u u_{x_1} - \xi_{2,x_1} u_{x_2} - \xi_{1,x_1} u_{x_1} - \xi_{3,x_1} u_{x_3} - \xi_{1,u} u_{x_1^2} - \xi_{2,u} u_{x_1} u_{x_2}\\
&\quad - \xi_{3,u} u_{x_1} u_{x_3},
\end{aligned} \tag{7}$$

$$\begin{aligned}
\eta_2^{(1)} &= U_{x_2} + U_u u_{x_2} - \xi_{3,x_2} u_{x_3} - \xi_{2,x_2} u_{x_2} - \xi_{1,x_2} u_{x_1} - \xi_{1,u} u_{x_1} u_{x_2} - \xi_{2,u} u_{x_2}^2\\
&\quad - \xi_{3,u} u_{x_2} u_{x_3},
\end{aligned} \tag{8}$$

$$\begin{aligned}
\eta_3^{(1)} &= U_{x_3} + U_u u_{x_3} - \xi_{1,x_3} u_{x_1} - \xi_{3,x_3} u_{x_3} - \xi_{2,x_3} u_{x_2} - \xi_{1,u} u_{x_1} u_{x_3} - \xi_{2,u} u_{x_3} u_{x_2}\\
&\quad - \xi_{3,u} u_{x_3^2},
\end{aligned} \tag{9}$$

and $\eta_{i_1 i_2}^{(j)}$ by

$$\begin{aligned}
\eta_{11}^{(2)} &= U_{x_1 x_1} + 2U_{ux_2} u_{x_2} + U_{uu} u_{x_2}^2 + U_u u_{x_2 x_2} - \xi_{1,x_2 x_2} u_{x_1} - 2\xi_{1,ux_2} u_{x_1} u_{x_2} - 2\xi_{1,x_2} u_{x_1 x_2}\\
&\quad - \xi_{1,uu} u_{x_2}^2 u_{x_1} - \xi_{1,u} u_{x_2 x_2} u_{x_1} - 2\xi_{1,u} u_{x_2} u_{x_1 x_2} - \xi_{2,x_2 x_2} u_{x_x 2} - 2\xi_{2,ux_2} u_{x_2}^2 - 2\xi_{2,x_2} u_{x_2 x_2}
\end{aligned}$$

$$
\begin{aligned}
& -\xi_{2,uu}u_{x_2}^3 - 3\xi_{2,u}u_{x_2x_2}u_{x_2} - \xi_{3,x_2x_2}u_{x_3} - 2\xi_{3,ux_2}u_{x_3}u_{x_2} - 2\xi_{3,x_2}u_{x_3x_2} \\
& -\xi_{3,uu}u_{x_2}^2u_{x_3} - \xi_{3,u}u_{x_2x_2}u_{x_3} - 2\xi_{2,u}u_{x_2}u_{x_2x_3}, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\eta_{12}^{(2)} = {} & U_{x_1x_2} + U_{ux_2}u_{x_1} + U_{ux_1}u_{x_2} + U_{uu}u_{x_1}u_{x_2} + U_u u_{x_1x_2} - \xi_{1,x_1x_2}u_{x_1} - \xi_{1,ux_2}u_{x_1}^2 - \xi_{1,x_2}u_{x_1x_1} \\
& -\xi_{1,uu}u_{x_2}u_{x_1}^2 - \xi 1, ux_1u_{x_1}u_{x_2} - \xi_{1,u}u_{x_1x_2}u_{x_1} - \xi_{1,u}u_{x_2}u_{x_1x_1} - \xi_{2,x_1x_2}u_{x_2} \\
& -\xi_{2,ux_2}u_{x_2}u_{x_1} - \xi_{2,x_2}u_{x_1x_2} - \xi_{2,uu}u_{x_1}u_{x_2}^2 - \xi_{2,ux_1}u_{x_2}^2 - 2\xi_{2,u}u_{x_1x_2}u_{x_2} - \xi_{3,x_1x_2}u_{x_3} \\
& -\xi_{3,ux_2}u_{x_3}u_{x_1} - \xi_{3,x_2}u_{x_3x_t} - \xi_{3,uu}u_{x_1}u_{x_2}u_{x_3} - \xi_{3,ux_1}u_{x_2}u_{x_3} - \xi_{3,u}u_{x_1x_2}u_{x_3} \\
& -\xi_{3,u}u_{x_2}u_{x_1x_3} - \xi_{1,x_1}u_{x_1x_2} - \xi_{2,x_1}u_{x_2x_2} - \xi_{3,x_1}u_{x_3x_2} - \xi_{1,u}u_{x_1}u_{x_1x_2} - \xi_{2,u}u_{x_1}u_{x_2x_2} \\
& -\xi_{3,u}u_{x_1}u_{x_3x_2}, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
\eta_{13}^{(2)} = {} & U_{x_1x_3} + U_{ux_3}u_{x_1} + U_{ux_1}u_{x_3} + U_{uu}u_{x_1}u_{x_3} + U_u u_{x_1x_3} - \xi_{1,x_1x_3}u_{x_1} - \xi_{1,ux_1}u_{x_1}u_{x_3} - \xi_{1,x_3}u_{x_1x_1} \\
& -\xi_{1,uu}u_{x_3}u_{x_1}^2 - \xi 1, ux_3u_{x_1}^2 - 2\xi_{1,u}u_{x_1x_3}u_{x_1} - \xi_{2,x_1x_3}u_{x_2} - \xi_{2,ux_3}u_{x_2}u_{x_1} - \xi_{2,x_3}u_{x_1x_2} \\
& -\xi_{2,uu}u_{x_1}u_{x_2}u_{x_3} - \xi_{2,ux_1}u_{x_2}u_{x_3} - \xi_{2,u}u_{x_1x_2}u_{x_3} - \xi_{2,u}u_{x_2}u_{x_1x_3} - \xi_{3,x_1x_3}u_{x_3} \\
& -\xi_{3,ux_3}u_{x_3}u_{x_1} - \xi_{3,x_3}u_{x_3x_t} - \xi_{3,uu}u_{x_1}u_{x_3}^2 - \xi_{3,ux_1}u_{x_3}^2 - \xi_{3,uu}u_{x_1}u_{x_3}^2 \\
& -2\xi_{3,u}u_{x_3}u_{x_1x_3} - \xi_{1,x_1}u_{x_1x_3} - \xi_{2,x_1}u_{x_2x_3} - \xi_{3,x_1}u_{x_3x_3} - \xi_{2,u}u_{x_1}u_{x_2x_3} \\
& -\xi_{3,u}u_{x_1}u_{x_3x_3}, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\eta_{22}^{(2)} = {} & U_{x_2x_2} + 2U_{ux_2}u_{x_2} + U_{uu}u_{x_2^2} + U_u u_{x_2x_2} - \xi_{1,x_2x_2}u_{x_1} - 2\xi_{1,ux_2}u_{x_1}u_{x_2} - 2\xi_{1,x_2}u_{x_1x_2} \\
& -\xi_{1,uu}u_{x_2}^2u_{x_1} - \xi_{1,u}u_{x_2x_2}u_{x_1} - 2\xi_{1,u}u_{x_2}u_{x_1x_2} - \xi_{2,x_2x_2}u_{x_2} - 2\xi_{2,ux_2}u_{x_2}^2 - 2\xi_{2,x_2}u_{x_2x_2} \\
& -\xi_{2,uu}u_{x_2}^3 - 3\xi_{2,u}u_{x_2x_2}u_{x_2} - \xi_{3,x_2x_2}u_{x_3} - 2\xi_{3,ux_2}u_{x_3}u_{x_2} \\
& -2\xi_{3,x_2}u_{x_2x_3} - \xi_{3,uu}u_{x_2^2}u_{x_3} - \xi_{3,u}u_{x_2x_2}u_{x_3} - 2\xi_{3,u}u_{x_2}u_{x_2x_3}, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\eta_{23}^{(2)} = {} & U_{x_2x_3} + U_{ux_2}u_{x_3} + U_{ux_3}u_{x_2} + U_{uu}u_{x_2}u_{x_3} + U_u u_{x_2x_3} - \xi_{1,x_2x_3}u_{x_1} - \xi_{1,ux_2}u_{x_1} \\
& -\xi_{1,x_2}u_{x_1x_3} - \xi_{1ux_2}u_{x_3}u_{x_1} - \xi_{1,uu}u_{x_3}u_{x_1}u_{x_2} - \xi 1, ux_3u_{x_1}u_{x_2} - \xi_{1,u}u_{x_2x_3}u_{x_1} \\
& -\xi_{1,u}u_{x_2}u_{x_1x_3} - \xi_{2,x_2x_3}u_{x_2} - \xi_{2,ux_2}u_{x_2}u_{x_3} - \xi_{2,x_2}u_{x_3x_2} \\
& -\xi_{2,uu}u_{x_3}u_{x_2}^2 - \xi_{2,ux_3}u_{x_2}^2 - 2\xi_{2,u}u_{x_3x_2}u_{x_2} - \xi_{2,u}u_{x_3}u_{x_2x_2} - \xi_{3,x_2x_3}u_{x_3} \\
& -\xi_{3,ux_2}u_{x_3}^2 - \xi_{3,x_2}u_{x_3x_3} - \xi_{3,uu}u_{x_2}u_{x_3}^2 - \xi_{3,ux_3}u_{x_3}u_{x_2} - \xi_{3,u}u_{x_2}u_{x_3x_3} \\
& -2\xi_{3,u}u_{x_3}u_{x_2x_3} - \xi_{1,x_3}u_{x_1x_2} - \xi_{2,x_3}u_{x_2x_2} - \xi_{3,x_3}u_{x_2x_3} - \xi_{2,u}u_{x_3}u_{x_2x_1}, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\eta_{33}^{(2)} = {} & U_{x_3x_3} + 2U_{ux_3}u_{x_3} + U_{uu}u_{x_3}^2 + U_u u_{x_3x_3} - \xi_{1,x_3x_3}u_{x_1} - 2\xi_{1,ux_3}u_{x_1}u_{x_3} - 2\xi_{1,x_3}u_{x_1x_3} \\
& -\xi_{1,uu}u_{x_3}^2u_{x_1} - \xi_{1,u}u_{x_3x_3}u_{x_1} - 2\xi_{1,u}u_{x_3}u_{x_1x_3} - \xi_{2,x_3x_3}u_{x_2} - \xi_{2,ux_3}u_{x_2}u_{x_3} - 2\xi_{2,x_3}u_{x_2x_3} \\
& -\xi_{2,ux_3}u_{x_2x_3} - \xi_{2,uu}u_{x_3}^2u_{x_2} - \xi_{2,u}u_{x_3x_3}u_{x_2} - \xi_{2,u}u_{x_3}u_{x_2x_3} - \xi_{3,x_3x_3}u_{x_3} - 2\xi_{3,ux_3}u_{x_3}^2 \\
& -2\xi_{3,x_3}u_{x_3x_3} - \xi_{3,uu}u_{x_3}^3 - 3\xi_{3,u}u_{x_3x_3}u_{x_3}, \qquad (15)
\end{aligned}
$$

We then have

$$X^{(2)}F(x_1, x_2, x_3, u, u_1, u_2, u_3, u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0$$

If $u = \ominus(x_1, x_2, x_3)$ is an invariant solution of (3) corresponding to (5) admitted by PDE (3) then

3

$u = \ominus(x_1, x_2, x_3)$ is an invariant surface of (5) if $X(u - \ominus(x_1, x_2, x_3)) = 0$ when $u = \ominus(x_1, x_2, x_3)$ or

$$\sum_{i=1}^{3} \xi_i(x, \ominus(x_1, x_2, x_3)) \frac{\partial \ominus}{\partial x_i} = \eta(x_1, x_2, x_3, \ominus(x_1, x_2, x_3)), \tag{16}$$

And $u = \ominus(x_1, x_2, x_3)$ solves (3) if $F(x_1, x_2, x_3, u, u_1, u_2, u_3, u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0$.

Equation (16) is known as the $invariant\ \ surface\ \ condition$ for an invariant solutions corresponding to (5). We obtain the invariant solutions using the invariant form method described below:

Lie's Classical Method is applied to a single PDE (2) and we obtain Lie algebra of infinitesimal symmetries is spanned by the six vector fields.

2. Lie's Classical Method to single PDE

We now seek the following group of infinitesimal transformations

$$\begin{aligned} w^* &= w + \epsilon W(t, x, u), \\ t^* &= t + \epsilon T(t, x, y, u), \\ x^* &= x + \epsilon X(t, x, y, u), \\ y^* &= y + \epsilon Y(t, x, y, u) \end{aligned} \tag{17}$$

under which (2) is invariant. Then

$$\begin{aligned} & W_t + W_w w_t - T_t w_t - T_w w_t^2 - X_t w_x - X_w w_x w_t - Y_t w_y - Y_w w_t w_y \\ & +w_x \left[W_x + W_w w_x - T_x w_t - T_w w_t w_x - X_x w_x - X_w w_x^2 - Y_x w_y - Y_w w_x w_y\right] \\ & +W_{yy} + 2W_{wy} w_y + W_{ww} w_y^2 + W_w w_{yy} - T_{yy} w_t - 2T_{wy} w_t w_y - 2T_y w_{ty} - T_{ww} w_y^2 w_t \\ & -T_w w_{yy} w_t - T_w w_y w_{ty} - X_{yy} w_x - X_{wy} w_x w_y - 2X_y w_{xy} - X_{wy} w_{xy} - X_{ww} w_y^2 w_x \\ & -X_w w_{yy} w_x - X_w w_y w_{xy} - Y_{yy} w_y - 2Y_{wy} w_y^2 - Y_{ww} w_y^3 - 2Y_w w_y w_{yy} - T_w w_y w_{ty} \\ & -X_w w_y w_{xy} - 2Y_y w_{yy} - Y_w w_y w_{yy} = 0. \end{aligned} \tag{18}$$

Now replacing the highest derivative term w_{yy} using (2), equation (18) becomes

$$\begin{aligned} & W_t + W_w w_t - T_t w_t - T_w w_t^2 - X_t w_x - X_w w_x w_t - Y_t w_y - Y_w w_t w_y \\ & +w_x \left[W_x + W_w w_x - T_x w_t - T_w w_t w_x - X_x w_x - X_w w_x^2 - Y_x w_y - Y_w w_x w_y\right] \\ & +W_{yy} + 2W_{wy} w_y + W_{ww} w_y^2 - T_{yy} w_t - 2T_{wy} w_t w_y - 2T_y w_{ty} - T_{ww} w_y^2 w_t \end{aligned}$$

4

$$
\begin{aligned}
-T_w w_{yy} w_t - T_w w_y w_{ty} - X_{yy} w_x - X_{wy} w_x w_y - 2X_y w_{xy} - X_{wy} w_{xy} - X_{ww} w_y^2 w_x \\
-X_w w_y w_{xy} - Y_{yy} w_y - 2Y_{wy} w_y^2 - Y_{ww} w_y^3 - T_w w_y w_{ty} - X_w w_y w_{xy} \\
- [W_w - X_w w_x - 2Y_w w_y - 2Y_y - Y_w w_y] \left(w_t + \frac{1}{2} w_x^2 \right) &= 0. \qquad (19)
\end{aligned}
$$

Equating the coefficients of $w_t^2, w_x w_t, w_t w_y, w_{ty}, w_{xy}$ to zero, we obtain

$$
\begin{aligned}
T_w &= 0 \\
X_w &= 0 \\
Y_w &= 0 \\
T_x &= 0 \\
Y_x &= 0 \\
T_y &= 0 \\
X_y &= 0.
\end{aligned}
$$

Again equating the derivativess of $w_y^2, w_x, w_x^2, w_t, w_y$ to zero, we obtain the six linear partial differential equations,

$$
\begin{aligned}
W_{ww} &= 0, & (20) \\
W_x - X_t &= 0, & (21) \\
\frac{1}{2} W_w - X_x + Y_y &= 0, & (22) \\
W_t + W_{yy} &= 0, & (23) \\
2Y_y - T_t &= 0, & (24) \\
2W_{wy} - Y_t - Y_{yy} &= 0. & (25)
\end{aligned}
$$

Integrating equation (20), we obtain

$$W_w = k_1(x, t, y). \qquad (26)$$

Again integrating equation (26),

$$W = k_1(x, t, y) w + k_2(x, t, y). \qquad (27)$$

Integrating with respect to y, equation (24) yields

$$Y = \frac{1}{2} T_t y + h_1(t). \qquad (28)$$

Putting equation (28) into equation(25) and integrating with respect to y, we arrive at

$$k_1 = \frac{1}{2} \left[\frac{1}{4} y^2 T_{tt} + h_1' y + k_3(x, t) \right]. \qquad (29)$$

5

Substituting (27) in (28) and integrating with respect to x we get

$$X = \frac{1}{2}\left[\frac{1}{8}T_{tt}y^2 + T_t\right]x + \frac{1}{4}xyh_1' + \frac{1}{4}\int k_3(x,t)dx + X_1(t). \tag{30}$$

Inserting equations (30) and (27) into equation (21), we obtain $wk_{1,x} + k_{2,x} = X_t$. But we find that $X = X(x,t)$. Therefore

$$k_{1,x} = 0, \quad X_t = k_{2,x}. \tag{31}$$

Using (30) and integrating with respect to x, equation (31) becomes

$$k_2 = \frac{x^2}{4}\left[\frac{1}{8}T_{ttt}y^2 + T_{tt}\right] + \frac{xy}{4}h_1'' + \frac{1}{4}\int k_3dt + xX_1(t) + H_1(y,t), \tag{32}$$

where $H_1(y,t)$ is a constant of integration. Substituting (27), (31) and (32) in equation (23) we get

$$wk_{1,t} + k_{2,t} + wk_{1,yy} + k_{2,yy} = 0. \tag{33}$$

Now collecting the different powers of w, we get two equations,

$$k_{1,t} + k_{1,yy} = 0, \tag{34}$$
$$k_{2,t} + k_{2,yy} = 0, \tag{35}$$

Using (32) equation (35) reduces to

$$\frac{x^2}{4}\left[\frac{1}{8}T_{tttt}y^2 + T_{ttt}\right] + \frac{x^2y}{8}h_1''' + \frac{1}{4}k_3(t) + xX_1(tt) + H_{1,t} + \frac{x^2}{16}T_{ttt} + H_{1,yy} = 0. \tag{36}$$

Now equating the different powers of x to zero , we get

$$\frac{1}{32}T_{tttt}y^2 + \frac{y}{8}h_1''' + \frac{5}{16}T_{ttt} = 0, \tag{37}$$
$$X_{1,tt} = 0, \tag{38}$$
$$\frac{k_3}{4} + H_{1,t} + H_{1,yy} = 0. \tag{39}$$

Solving equations (37) and (38) we obtain that

$$\begin{aligned} T &= \frac{a}{2}t^2 + bt + c, \\ h_1 &= \frac{h_2}{2}t^2 + h_3t + h_4, \\ X_1 &= d_1t + d_2. \end{aligned} \tag{40}$$

Substituting equations (29) and (40) in equation (34) we get

$$\frac{1}{2}h_2 y + k_{3,t} + \frac{a}{4} = 0. \tag{41}$$

Clearly we conclude that

$$\begin{aligned} h_2 &= 0 \\ k_3 &= -\frac{a}{4}t + a_0. \end{aligned} \tag{42}$$

Putting (40), (42) into equations (30) and (32), we find that

$$X = \frac{1}{2}\left[\frac{a}{8}y^2 + at + b\right]x + \frac{h_3}{4}xy + \frac{1}{4}\left[-\frac{a}{4}t + a_0\right]x + d_1 t + d_2 + a_2, \tag{43}$$

$$k_2 = \frac{a}{4}x^2 + \frac{1}{4}\left[-\frac{a}{8}t^2 + a_0 t + a_1\right] + d_1 x + H_1(y,t). \tag{44}$$

Substituting (43)-(44) in equation (31), we conclude that a must be zero.

Inserting $a = 0$ and equations (44), (42)into equations (43), (40), (28) and (27) , finally we get

$$T = bt + c, \tag{45}$$

$$Y = \frac{b}{2}y + h_3 t + h_4, \tag{46}$$

$$X = \frac{b}{2}x + \frac{h_3}{4}xy + d_1 t + d_2 + \frac{a_0}{4}x + a_4, \tag{47}$$

$$W = \left[\frac{h_3}{4}y + \frac{a)}{2}\right]w + \frac{1}{4}(a_0 t + a_1) + d_1 x + H_1(y,t), \tag{48}$$

with $\frac{a_0}{4} + H_{1,t} + H_{1,yy} = 0$.

Special Case :

For $H_1 = k_3 = 0$ from (45)-(48), we obtain

$$T = bt + c, \tag{49}$$

$$Y = \frac{b}{2}y + h_3 t + h_4, \tag{50}$$

$$X = \frac{b}{2}x + \frac{h_3}{4}xy + d_1 t + d_2, \tag{51}$$

$$W = \frac{h_3}{4}yw + d_1 x. \tag{52}$$

Consequently,

$$\begin{aligned}
X &= X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + Y\frac{\partial}{\partial y} + W\frac{\partial}{\partial w}, \\
&= \left[\frac{b}{2}x + \frac{h_3}{4}xy + d_1t + d_2\right]\frac{\partial}{\partial x} + [bt + c]\frac{\partial}{\partial t} \\
&\quad + \left[\frac{b}{2}y + h_3t + h_4\right]\frac{\partial}{\partial y} + \left[\frac{h_3}{4}yw + d_1x\right]\frac{\partial}{\partial w}, \\
&= b\left[\frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{y}{2}\frac{\partial}{\partial y}\right] + h_3\left[\frac{xy}{4}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{yw}{2}\frac{\partial}{\partial w}\right] \\
&\quad + d_1\left[t\frac{\partial}{\partial x} + x\frac{\partial}{\partial w}\right] + d_2\frac{\partial}{\partial x} + c\frac{\partial}{\partial t} + h_4\frac{\partial}{\partial y}. \qquad (53)
\end{aligned}$$

Thus the Lie algebra of infinitesimal symmetries of (2) is spanned by the six vector fields.

$$\begin{aligned}
X_1 &= \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{y}{2}\frac{\partial}{\partial y}, \\
X_2 &= \frac{xy}{4}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{yw}{2}\frac{\partial}{\partial w}, \\
X_3 &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial w}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = \frac{\partial}{\partial t}, \quad X_6 = \frac{\partial}{\partial y}, \qquad (54)
\end{aligned}$$

The commutation relations between these vector fields are given by the following table. The entry in row i and column j representing $[X_i, X_j]$:

Table

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$-\frac{1}{2}X_4$	$-X_5$	$-\frac{1}{2}X_6$
X_2	0	0	0	0	0	0
X_3	0	0	0	0	0	0
X_4	$\frac{1}{4}X_4$	0	0	0	0	0
X_5	X_5	0	0	0	0	0
X_6	$\frac{1}{2}X_6$	0	0	0	0	0

The corresponding local one parameter groups are the following :

$$G_1 \; : \quad (u, x, y, t) \quad (\sigma_s^1) \to \quad \left(\frac{x}{2}\exp s, t\exp s, \frac{y}{2}\exp s\right),$$

8

$$
\begin{array}{llll}
G_2 & : & (u,x,y,t) & (\sigma_s^2) \rightarrow & \left(\frac{xy}{4}\exp s, t\exp s, \frac{yw}{2}\exp s\right), \\
G_3 & : & (u,x,y,t) & (\sigma_s^3) \rightarrow & \left(t\exp s, x\exp s\right), \\
G_4 & : & (u,x,y,t) & (\sigma_s^4) \rightarrow & \left(w, x+s, y, t\right), \\
G_5 & : & (u,x,y,t) & (\sigma_s^5) \rightarrow & \left(w, x, y, t+s\right), \\
G_6 & : & (u,x,y,t) & (\sigma_s^6) \rightarrow & \left(w, x, y+s, t\right).
\end{array} \quad (55)
$$

where $\sigma_s^j = exp(sX_j)$, $1 \leq j \leq 6$. Since each local Lie group G_j is a symmetry group, exponentiation shows that if $w = w(x,y,t)$ is a solution of (2) then so are

$$
\begin{aligned}
w^1(x,y,t) &= w\left(-\frac{x}{2}\exp(-s), t\exp(-s), \frac{y}{2}\exp(-s)\right), \\
w^2(x,y,t) &= w\left(\frac{xy}{4}\exp(-s), t\exp(-s), \frac{yw}{2}\exp(-s)\right), \\
w^3(x,y,t) &= w\left(t\exp(-s), x\exp(-s)\right), \\
w^4(x,y,t) &= w\left(x-s, y, t\right), \\
w^5(x,y,t) &= w\left(x, y, t-s\right), \\
w^6(x,y,t) &= w\left(x, y-s, t\right).
\end{aligned} \quad (56)
$$

3. CONCLUSION : In this paper we found new symmeries for the equation(2) and the infinite group of point transformations are invariant. And also we found the corresponding local one parameter groups (55). The Lie algebra of infinitesimal symmetries of (2) is spanned by the six vector fields eqn (54). The corresponding local one parameter groups are obtained eqn (55). And also each local Lie group G_j is a symmetry group, exponentiation shows that if $w = w(x,y,t)$ is a solution of (2) then so are eqn(56).

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