Generalized Ulam-Hyers Stability of two types of n-dimensional Quadratic functional equation in Banach Space: Direct and Fixed Point Methods

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Abstract. *In this paper, the authors discussed the generalized UH stability of two types of n-dimensional quadratic functional equation of the form*

$$f\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1}^{n} f\left(-x_j+\sum_{i=1, i\neq j}^{n} x_i\right)$$

$$=(n-3)\sum_{i,j=1, 1\leq i<j\leq n} f\left(x_i+x_j\right)+\left(-n^2+5n-2\right)\sum_{i=1}^{n} f\left(x_i\right)$$

where n *is a positive integer with* $n\geq 3$, *in Banach Space using direct and fixed point methods.*

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1. INTRODUCTION AND PRELIMINARIES

S.M.Ulam, in this famous lecture in 1940 to the mathematics club of university as the stability problems of Wisconsin, presented a number of unsolved problems. This is the starting point of the theory of the stability of functional equations. One of the questions led to a new line of investigation, nowadays known as the stability problems.

For very general functional equations one can ask the following question .When is it it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? Similarly, if we replace a given functional inequality, when can assert that the solutions of the inequality lie near to the solutions of the strict equation?

Suppose that G is a group, $H(d)$ is a metric group, and $f:G\rightarrow H$,for any $\in>0$,does there exists a $\delta>0$ such that

$$d\left(f(xy), f(x)f(y)\right)<\in$$

Holds for all $x,y\in G$?

These kinds of the questions from the basis of stability theory, and D.H.Hyers [14,15] obtained the first important result in this field .Many examples of this have been solved and many variations have been studied since.

The quadratic function $f(cx)=cx^2$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

And therefore the equation (1.1) is called the quadratic functional equation.

The Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved by F.skof for the functions $f:E_1\rightarrow E_2$ where E_1 is the normed space and E_2 be a Banach space, the result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group it was delt by P.W.Choelewa [10].S.Czerwik [11,12] proved the Hyer-Ulam-Rassias stability of the quadratic functional equation.This result further generalized by Th.M.Rassias [40-45].C.Borelli ,and G.L.Forti [9]

In 2006,K.W.Jun and H.M.Kim [16-20] introduced the following generalized AQ type functional equation

$$f\left(\sum_{i=1}^{n} x_i\right)+(n-2)\sum_{i=1}^{n} f\left(x_i\right)=\sum_{1\leq i<j\leq n} f\left(x_i+x_j\right) \tag{1.2}$$

in the class of functional equation between real vector spaces. For $n=3$,Pl.kannappan[33] proved that a function f satisfies the functional equation (1.2) if and only if there exists a symmetric bi-additive function B such that $f(x)=B(x,x)+A(x)$ for all (see[18]).The Hyers –Ulam stability for the equation (1.2) when $n=3$ was proved by S.M.Jung [20] .The Hyers-Ulam- Rassias for the equation (2) when $n=4$ was also investigated by I.S.Chang et al.,[6] Recently,M.Arumkumar ,S.Murthy and G.Ganapathy introduced the general solution and generalized Ulam-Hyers stability of n-dimensional quadratic functional of the form

$$\sum_{i=1}^{n} g\left(\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n}(n-i+1)g(x_i) = \frac{1}{2}\sum_{i=1}^{n-1}(n-1)\sum_{j=1}^{n} g(x_j+x_{i+1}) - g(x_j-x_{i+1})$$

(1.3)

In Banach spaces.

Very recently, J.M.Rassias introduced the Leibnitz type additive quadratic functional equation of the form

$$f(x-t)+f(y-t)+f(z-t) = f\left(\frac{x+y+z}{3}-t\right) f\left(\frac{2x-y-z}{3}\right) f\left(\frac{-x+2y-z}{3}\right) f\left(\frac{-x-y+2z}{3}\right)$$

(1.4)

And obtained its general solution and generalized Ulam-Hyers-stability of Leibnitz AQ-mixed type functional equation in Quasi-Beta normed space using direct and fixed point methods .

In this paper,the authors obtain the generalized Ulam-Hyers stability of n dimensional quadratic functional

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i\right) = (n-3)\sum_{i,j=1, 1\le i<j\le n} f\left(x_i+x_j\right) + \left(-n^2+5n-2\right)\sum_{i=1}^{n} f\left(x_i\right)$$

(1.5)

Where n is a positive integer with $n\ge 3$ in Banach spaces using Direct and fixed point methods.

2. STABILITY RESULTS – DIRECT METHOD

In this section, we present the generalized Ulam-Hyers stability of the functional equation (1.5)

Theorem 2.1. Let $j\in\{-1,1\}$ and $\alpha: X^n \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha\left(2^{kj}x_1, 2^{kj}x_2, \ldots, 2^{kj}x_n\right)}{4^{kj}}$$

Converges in R and

$$\lim_{k\to\infty} \frac{\alpha\left(2^{kj}x_1, 2^{kj}x_1, \ldots, 2^{kj}x_n\right)}{4^{kj}} = 0 \quad (2.1)$$

For all $x,y,z\in X$. Let $f: X\to Y$ be an even function satisfying the inequality

$$\| Df(x_1,x_2,\ldots,x_n) \| \le \alpha(x,-x,x,-x,x,0,\ldots,0)$$

(2.2)

for all $x_1,x_2,\ldots,x_n \in X$. There exists a unique quadratic mapping $Q: X\to Y$ which satisfies the functional equation (1.5) and

$$\| f(x)-Q(x) \| \le \frac{1}{8(n-5)}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\eta\left(2^{kj}x\right)}{4^{kj}}$$

(2.3)

Where $\eta(x)=\alpha(x,-x,x,-x,x,0,\ldots,0)$ for all $x\in X$. The mapping $Q(x)$ is defined

By $$Q(x)=\lim_{k\to\infty}\frac{f\left(2^{kj}x\right)}{4^{kj}} \quad (2.4)$$

for all $x\in X$.

Proof. Assume that $j=1$. Replacing $(x_1,x_1,\ldots,x_n)$ and $(x,-x,x,-x,x,0,\ldots,0)$ in (2.2) ,we get

$$\| 2(n-5)f(2x)-8(n-5)f(x) \| \le \alpha(x,-x,x,-x,x,0,\ldots,0)$$

(2.5)

for all $x\in X$. It follows from (2.5) that

$$\left\| \frac{f(2x)}{4} - f(x) \right\| \le \frac{1}{8(n-5)}\eta(x)$$

(2.6)

Where $\eta(x)=\alpha(x,-x,x,-x,x,0,\ldots,0)$ for all $x\in X$. Replacing x by $2x$ in (2.6) and dividing by 4 and adding the resultant inequality with (2.6), we obtain

$$\left\| \frac{f(2^2x)}{4^2} - f(2x) \right\| \le \frac{1}{8(n-5)}\left[\eta(x)+\frac{\eta(2x)}{4}\right]$$

(2.7)

for all $x\in X$. Generalizing, we have

$$\left\| f(x)-\frac{f(2^k x)}{4^k} \right\| \le \frac{1}{8(n-5)} \sum_{k=0}^{n-1} \frac{\eta(2^k x)}{4^k}$$
$$\le \frac{1}{8(n-5)} \sum_{k=0}^{\infty} \frac{\eta(2^k x)}{4^k}$$
(2.8)

for all $x \in X$. In order to prove convergence of the sequence $\left\{\frac{f(2^k x)}{4^k}\right\}$,replace x by $2^l x$ and dividing 2^l (2.8), for any $k,l>0$, to deduce

$$\left\| \frac{f(2^l x)}{4^l} - \frac{f(2^{k+l} x)}{4^{k+l}} \right\| = \frac{1}{4^l} \left\| f(2^l x) - \frac{f(2^k.2^l x)}{4^k} \right\|$$
$$\le \frac{1}{8(n-5)} \sum_{k=0}^{n-1} \frac{\eta(2^{k+l} x)}{4^{k+l}}$$
(2.9)
$$\le \frac{1}{8(n-5)} \sum_{k=0}^{\infty} \frac{\eta(2^{k+l} x)}{4^{k+l}}$$
$$\to 0 \text{ as } l \to \infty$$
for all $x \in X$.

Hence the sequence $\left\{\frac{f(2^k x)}{4^k}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q: X \to Y$ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(2^k x)}{4^k}, \quad \forall\ x \in X.$$

Letting $k \to \infty$ in (2.8), we see that (2.4) holds for $x \in X$. To prove that Q satisfies (1.5) replacing $(x_1, x_2, ..., x_n)$ by $(2^k x_1, 2^k x_2, ..., 2^k x_n)$ and dividing 4^k in (2.2), we obtain

$$\frac{1}{4^k} \left\| Df(2^k x_1, 2^k x_2, ..., 2^k x_n) \right\| \le \frac{1}{4^k} \phi(2^k x_1, 2^k x_2, ..., 2^k x_n)$$

for all $x_1, x_2, ..., x_n \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that

$$DQ(x_1, x_2, ..., x_n) = 0.$$

Hence Q satisfies (1.5) for all $x_1, x_2, ..., x_n \in X$. To show that Q is unique, let $B(x)$ be another quadratic mapping satisfying (1.5) and (2.3), then

$$\| Q(x) - B(x) \| = \frac{1}{4^l} \| Q(2^l x) - B(2^l x) \|$$
$$\le \frac{1}{4^l} \| Q(2^l x) - f(2^l x) \| + \| f(2^l x) - B(2^l x) \|$$
$$\le \frac{1}{8(n-5)} \sum_{k=0}^{\infty} \frac{\eta(2^{k+l})}{4^{k+l}} \to 0 \text{ as } l \to \infty$$
for all $x \in X$. Hence Q is unique.

for all $x \in X$. The rest of the proof is similar to that of $j=1$. Hence for $j=-1$ also the theorem is true. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.5).

Corollary 2.2. Let μ and s be a nonnegative real numbers. Let an even function $f: X \to Y$ satisfying the inequality

$$\left\| f(x_1, x_2, \cdots, x_n) \right\| \le \begin{cases} \mu, \\ \mu \left\{ \sum_{k=1}^{n} \|x_k\|^s \right\} \\ \mu \left\{ \prod_{k=1}^{n} \|x_k\|^s + \sum_{k=1}^{n} \|x_k\|^{ns} \right\}, \end{cases}$$
(2.10)

for all $x_1, x_2, ..., x_n \in X$. Then there exists a unique quadratic function $Q: X \to Y$ such that

$$\| f(x) - Q(x) \| \le \begin{cases} \frac{\mu}{6|n-5|}, \\ \frac{5\mu \|x\|^s}{2(n-5)|4-2^s|}, \\ \frac{5\mu \|x\|^{3s}}{2(n-5)|4-2^{ns}|}, \end{cases}$$
(2.11)

for all $x \in X$.

Proof: If we replace

$$\phi(x_1, x_2, ..., x_n) \le \begin{cases} \mu, \\ \mu \left\{ \sum_{k=1}^{n} \|x_k\|^s \right\} \\ \mu \left\{ \prod_{k=1}^{n} \|x_k\|^s + \sum_{k=1}^{n} \|x_k\|^{ns} \right\}, \end{cases}$$
(2.12) in Theorem 2.1, we arrive (2.11).

3.FIXED POINT STABILITY RESULTS OF (1.5)

The following theorems are useful to prove our fixed point stability results.

Theorem A. (Banach contraction principle) Let (X,d) be a complete metric spaces and consider a mapping $T:X\to X$ which is strictly contractive mapping, that is,

(A_1) $d(Tx,Ty)\le d(x,y)$

for some (Lipschtiz constant) $L<1$, then,

(i) The mapping T has one and only fixed point $x^*=T(x^*)$.

(ii) The fixed point for each given element x^* is globally contractive that is

(A_2)

$$\lim_{n\to\infty}T^n x=x^*$$

for any starting point $x\in X$.

(iii) One has the following estimation inequalities,

(A_3)

$$d\left(T^n x,x^*\right)\le\frac{1}{1-L}d\left(T^n x,T^{n+1}x\right),\forall n\ge 0,\forall x\in X.$$

(A_4)

$$d\left(x,x^*\right)=\frac{1}{1-L}d\left(x,x^*\right),\qquad \forall x\in X.$$

Theorem B. (The Alternative Fixed Point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T:X\to X$ with Lipschtiz constant L, then for each given element $x\in X$ either,

(B_1)

$$d(T^n x,T^{n+1}x)=\infty,\qquad \forall n\ge 0.$$

(B_2) There exists a natural number n_0 such that,

(i) $d(T^n x,T^{n+1}x)<\infty$ for all $\forall n\ge 0$.

(ii) The sequence $\{T^n x\}$ is convergent to a fixed point y^* of T,

(iii) y^* is the unique fixed point of T in the set $Y=\left\{y\in Y; d(T^{n_0}x,y)<\infty\right\}$.

(iv) $d\left(y^*,y\right)\le\frac{1}{1-L}d\left(y,Ty\right)$ for all $y\in Y$.

4. Fixed Point Stability of (1.5):

In this section, we present the generalized Ulam-Hyers stability of the functional equation (1.5) using fixed point method.

Theorem 4.1. Let $f:V\to B$ be an even mapping for which there exists a function $\alpha:V^n\to[0,\infty)$ with the condition

$$\lim_{k\to\infty}\frac{\alpha\left(2^k x_1,2^k x_2,\ldots,2^k x_n\right)}{4^k}=0 \tag{4.1}$$

where

$$\eta_i=\begin{cases}2, & i=0;\\ \frac{1}{2}, & i=1;\end{cases}$$

such that the functional inequality

$$\| Df(x_1,x_2,\ldots,x_n)\|\le\alpha(x,-x,x,-x,x,0,\ldots,0) \tag{4.2}$$

for all $x_1,x_2,\ldots,x_n\in V$. If there exist $L=L(i)$ such that the function

$$x\to\beta(x)=\frac{1}{2(n-5)}\alpha\left(\frac{x}{2},-\frac{x}{2},\frac{x}{2},-\frac{x}{2},\frac{x}{2},0,\ldots,0\right)$$

has the property,

$$\frac{1}{\eta_i^2}\beta(\eta_i x)=L\beta(x) \tag{4.3}$$

for all $x\in V$. Then there exists a unique quadratic function $Q:V\to B$ satisfying the functional equation (1.5) and

$$\| f(x)-Q(x)\|\le\frac{L^{1-i}}{1-L}\beta(x) \tag{4.4}$$

holds for all $x\in V$.

Proof. Consider the set

$$X=\{P/P:V\to B, P(0)=0\}$$

and introduce the generalized metric on X.

$$d(p,q)=\inf\left\{K\in(0,\infty):\| p(x)-q(x)\|\le K\beta(x), x\in V\right\}.$$

It is easy to see that (X,d) is complete. Define $T:X\to X$ by

$$Tp(x)=\frac{1}{\eta_i^2}p(\eta_i x)$$

for all $x\in V$. Now $p,q\in X$

$$d(p,q) \le K \Rightarrow \| p(x) - q(x) \| \le K\beta(x), x \in V;$$

$$\Rightarrow \left\| \frac{1}{\eta_i^2} p(\eta_i x) - \frac{1}{\eta_i^2} q(\eta_i x) \right\| \le \frac{1}{\eta_i^2} K\beta(\eta_i x), x \in V;$$

$$\Rightarrow \left\| \frac{1}{\eta_i^2} p(\eta_i x) - \frac{1}{\eta_i^2} q(\eta_i x) \right\| \le LK\beta(x), x \in V$$

$$\Rightarrow \| Tp(x) - Tq(x) \| \le LK\beta(x), x \in V;$$

$$\Rightarrow d(Tp,Tq) \le LK.$$

This implies

$d(Tp,Tq) \le Ld(p,q)$ for all $p,q \in X$.

(i,e.,) T is strictly contractive mapping on X with Lipschtiz constant L. It is follows from (2.5) that

$$\| 2(n-5) f(2x) - 8(n-5) f(x) \| \le \alpha(x,-x,x,-x,x,0,\dots,0) \quad (4.5)$$

for all $x \in V$. It is follows from (4.5) that

$$\left\| f(x) - \frac{f(2x)}{4} \right\| \le \frac{1}{8(n-5)} \phi(x,-x,x,-x,x,0,\dots,0) \quad (4.6)$$

for all $x \in V$. Using (4.5), for the case $i = 0$, it reduces to

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \le \frac{1}{4}\beta(x) \quad (4.7)$$

for all $x \in V$.

(i.e.,) $d(f,Tf) \le \frac{1}{2}$

$$\Rightarrow d(f,Tf) \le \frac{1}{2} = L = L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (4.5), we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \le \frac{1}{2(n-5)} \phi\left(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, 0, \dots, 0\right) \quad (4.8)$$

for all $x \in V$. Using (4.3) for the case $i = 1$, it reduces to,

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \le \beta(x) \quad (4.9)$$

for all $x \in V$.

(i.e.,) $d(f,Tf) \le 1 \Rightarrow d(f,Tf) \le 1 = L^0 < \infty$.

In above case, we arrive

$$d(f,Tf) \le L^{1-i}$$

Therefore $(B_2(i))$ holds. By $(B_2(ii))$, it follows that there exists a fixed point Q of T in X, such that

$$Q(x) = \lim_{k\to\infty} \frac{f(2^k x)}{4^k}, \quad \forall x \in V. \quad (4.10)$$

In order to prove $Q: V \to B$ is quadratic. Replacing $(x_1, x_2, \dots, x_n)$ by $(2^k x_1, 2^k x_2, \dots, 2^k x_n)$ in (4.2) and dividing by 4^k, it follows from (4.1) and (4.10), we see that Q satisfies (1.5) for all $x_1, x_2, \dots, x_n \in V$. Hence Q satisfies the functional equation (1.5).

By $(B_2(iii))$, Q is the unique fixed point of T in the set,

$$Y = \{ f \in X : d(Tf, Q) < \infty \}$$

Using the fixed point alternative result, Q is the unique function such that,

$$\| f(x) - Q(x) \| \le K\beta(x)$$

for all $x \in V$, and $k > 0$.

Finally by $(B_2(iv))$, we obtain

$$d(f,Q) \le \frac{1}{1-L} d(f,Tf)$$

$$d(f,Q) \le \frac{L^{1-i}}{1-L}.$$

Hence, we conclude that

$$\| f(x) - Q(x) \| \le \frac{L^{1-i}}{1-L}\beta(x)$$

for all $x \in V$. This completes the proof of the theorem.

Corollary 7.2. Let $f: V \to B$ be an even mapping and there exists a real numbers μ and s such that,

$$\left\|f\left(x_1,x_2,\cdots,x_n\right)\right\| \le \begin{cases} \mu, \\ \mu\left\{\sum_{k=1}^{n}\left\|x_k\right\|^s\right\} \\ \mu\left\{\prod_{k=1}^{n}\left\|x_k\right\|^s+\sum_{k=1}^{n}\left\|x_k\right\|^{ns}\right\}, \end{cases}$$

(4.11)

for all $x_1,x_2,...,x_n \in V$. There exists a quadratic mapping $Q:V\to B$ such that

$$\| f(x)-Q(x) \| \le \begin{cases} \frac{\mu}{6\left|n-5\right|}, \\ \frac{5\mu\| x \|^s}{2(n-5)\left|4-2^s\right|}, \\ \frac{5\mu\| x \|^{ns}}{2(n-5)\left|4-2^{ns}\right|}, \end{cases}$$

(4.12)

for all $x\in V$.

Proof: Setting

$$\alpha\left(x_1,x_2,\cdots,x_n\right) \le \begin{cases} \mu, \\ \mu\left\{\sum_{k=1}^{n}\left\|x_k\right\|^s\right\} \\ \mu\left\{\prod_{k=1}^{n}\left\|x_k\right\|^s+\sum_{k=1}^{n}\left\|x_k\right\|^{ns}\right\}, \end{cases}$$

for all $x_1,x_2,...,x_n \in V$. Now

$$\frac{\alpha\left(\eta^K x_1,\eta^K x_2,...,\eta^K x_n\right)}{\eta^{2k}} = \begin{cases} \frac{\mu}{\eta^{2k}}, \\ \frac{\mu}{\eta^{2k}}\left\{\sum_{k=1}^{n}\left\|\eta_i^k x\right\|^s\right\} \\ \frac{\mu}{\eta^{2k}}\left\{\prod_{k=1}^{n}\left\|\eta_i^k x\right\|^s+\sum_{k=1}^{n}\left\|\eta_i^k x\right\|^{ns}\right\}; \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k\to\infty, \\ \to 0 \text{ as } k\to\infty, \\ \to 0 \text{ as } k\to\infty, \end{cases}$$

(4.13)

i.e., (4.1) is holds. But we have $\beta(x)=\frac{1}{2(n-5)}\alpha\left(\frac{x}{2},-\frac{x}{2},\frac{x}{2},-\frac{x}{2},\frac{x}{2},0,...,0\right)$. Hence

$$\beta(x)=\frac{1}{2(n-5)}\alpha\left(\frac{x}{2},-\frac{x}{2},\frac{x}{2},-\frac{x}{2},\frac{x}{2},0,...,0\right) = \begin{cases} \lambda \\ \frac{\lambda}{4^s}\| x \|^s \\ \frac{\lambda}{4^{ns}}\| x \|^{ns} \end{cases}$$

Also,

$$\frac{1}{\eta_i^2}\beta\left(\eta_i x\right) = \begin{cases} \frac{1}{\eta_i^2}.\frac{\lambda}{6(n-5)} \\ \frac{5\lambda}{\eta_i^2}\frac{\|\eta_i x\|^s}{2(n-5)} \\ \frac{5\lambda}{\eta_i^2}\frac{\|\eta_i x\|^{ns}}{2(n-5)} \end{cases} = \begin{cases} \eta_i^{-2}\beta(x) \\ \eta_i^{s-2}\beta(x) \\ \eta_i^{ns-2}\beta(x) \end{cases}$$

(4.14)

Hence the inequality (4.13) holds

either $L=2^{-2}$ for $s=0$ if $i=0$ and $L=\frac{1}{3^{-2}}$ for $s=0$ if $i=1$.

either $L=2^{s-2}$ for $s<1$ if $i=0$ and $L=\frac{1}{2^{s-2}}$ for $s>1$ if $i=1$.

either $L = 2^{ns-2}$ for $s<1$ if $i=0$ and $L = \frac{1}{2^{ns-2}}$ for $s>1$ if $i=1$.

Now from (4.13), we prove the following cases

Case: 1 $L = 2^{-2}, \quad i=0$

$$\| f(x)-Q(x) \| \le \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(2^{-2}\right)^{1-0}}{1-2^{-2}} \frac{\lambda}{2(n-5)} = \frac{\lambda}{6(n-5)}$$

. (4.15)

Case: 2 $L = \left(\frac{1}{2}\right)^{-2}, \quad i=1$

$$\| f(x)-Q(x) \| \le \frac{L^{1-i}}{1-L}\beta(x)$$

$$= \frac{\left(\frac{1}{2^{-2}}\right)^{1-1}}{1-\left(\frac{1}{2}\right)^{-1}} \frac{\lambda}{2(n-5)} = \frac{\lambda}{6(5-n)} .$$

(4.16)

Case: 3 $L = 2^{s-2}, \quad s<1, \quad i=0$

$$\| f(x)-Q(x) \| \le \frac{L^{1-i}}{1-L}\beta(x)$$

$$= \frac{\left(2^{s-2}\right)^{1-0}}{1-2^{s-2}} \frac{5\lambda}{2^s 2(n-5)} \| x \|^s .$$

$$= \frac{5\lambda \| x \|^s}{2(n-5)(4-2^s)}$$

(4.17)

Case: 4 $L = \left(\frac{1}{2}\right)^{s-2}, \quad s>1, \quad i=1$

$$\| f(x)-Q(x) \| \le \frac{L^{1-i}}{1-L}\beta(x)$$

$$= \frac{\left(2^{2-s}\right)^{1-1}}{1-2^{2-s}} \frac{\lambda}{2^s 2(n-5)} \| x \|^s \quad (4.18)$$

$$= \frac{5\lambda \| x \|^s}{2(n-5)(2^s-4)}$$

Case: 5 $L = 2^{3s-2}, \quad s<\frac{1}{2}, \quad i=0$

$$\| f(x)-Q(x) \| \le \frac{L^{1-i}}{1-L}\beta(x)$$

$$= \frac{\left(2^{ns-2}\right)^{1-0}}{1-2^{ns-2}} \frac{5\lambda}{2^{ns} 2(n-5)} \| x \|^{ns} .$$

$$= \frac{\lambda \| x \|^{ns}}{2(n-5)(4-2^{ns})}$$

(4.19)

Case: 6 $L = \left(\frac{1}{2}\right)^{3s-2}, \quad s>\frac{1}{2}, \quad i=1$

$$\| f(x)-Q(x) \| \le \frac{L^{1-i}}{1-L}\beta(x) = \frac{\left(2^{2-ns}\right)^{1-1}}{1-2^{2-ns}} \frac{5\lambda}{2^{ns} 2(n-5)} \| x \|^{ns} = \frac{\lambda \| x \|^{ns}}{2(n-5)(2^{ns}-4)}$$

. (4.20)

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