Deriving Shape Functions for Hexahedral Element by Natural Coordinate System and Verified

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Abstract — *In this paper, I derived shape functions for hexahedral element by using natural Co-ordinate system and also I verified two verification conditions for shape functions. First verification condition is sum of all the shape functions is equal to one and second verification condition is each shape function has a value of one at its own node and zero at the other nodes. For computational purpose I used Mathematica 9 Software [2].*

Keywords — *Hexahedral element, Natural Co-ordinate system, Shape functions.*

I. INTRODUCTION

For any geometry to analyse heat and mass transfer first we should find out shape functions. A natural coordinate system is a coordinate system which permits the specification of a point within the element by a set of dimensionless numbers, whose magnitude never exceeds unity. It is obtained by assigning weightages by the nodal coordinates in defining the coordinate of any point inside the element. Hence such system has the property that ith coordinate has unit value at node i of the element and zero value at all other nodes.

II. GEOMETRICAL DESCREPTION

Hexahedral element with eight nodes is shown in figure.1 with axis ξ, η, ς respectively.

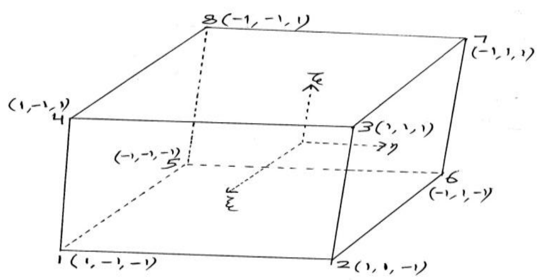

Figure.1 Typical hexahedron element

III.DERIVING SHAPE FUNCTIONS FOR HEXAHEDRAL ELEMENT

The typical element is shown in Figure.1

The natural coordinates of various nodal points are
1(1,-1,-1), 2(1, 1, -1), 3(1, 1, 1), 4(1, -1, 1),
5(-1, -1, -1), 6(-1, 1, -1), 7(-1, 1, 1), 8(-1, -1, 1)

There are only eight nodal values for defining displacement inside the element. Hence polynomial with 8 constants is to be selected for shape function. Keeping in view, geometric isotropy is to be maintained the following polynomial is selected.

$$u = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\varsigma + \alpha_5\xi\eta + \alpha_6\eta\xi + \alpha_7\varsigma\xi + \alpha_8\xi\eta\varsigma \qquad (1)$$

$$\overset{\xi\ \eta\ \varsigma}{1(1,-1,-1)} \quad \overset{\xi\ \eta\ \varsigma}{2(1,1,-1)} \quad \overset{\xi\ \eta\ \varsigma}{3(1,1,1)},$$

$$\overset{\xi\ \eta\ \varsigma}{4(1,-1,1)}, \quad \overset{\xi\ \eta\ \varsigma}{5(-1,-1,-1)}, \quad \overset{\xi\ \eta\ \varsigma}{6(-1,1,-1)},$$

$$\overset{\xi\ \eta\ \varsigma}{7(-1,1,1)}, \quad \overset{\xi\ \eta\ \varsigma}{8(-1,-1,1)}$$

$$\{u\}_e = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \varsigma & \xi\eta & \eta\varsigma & \varsigma\xi & \xi\eta\varsigma \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}$$

$$\{u\}_e = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} \quad (2)$$

$$\{u\}_e = [A]\{\alpha\} \quad (3)$$

$$\{\alpha\} = \frac{1}{[A]}\{u\}_e$$

$$\{\alpha\} = [A]^{-1}\{u\}_e \quad (4)$$

Where $\{\mathrm{u}\}_e$ is the vector of nodal displacements in x directions,
$[\mathrm{A}]$ is the matrix shown in eq(3) and $\{\alpha\}$ is the vector of generalized coordinates (constants in polynomials)

By using Mathematica Software calculate Inverse of A.

$$A := \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Inverse of [A]//MatrixForm

$$[A]^{-1} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

But $\mathrm{u} = [1 \;\; \xi \;\; \eta \;\; \varsigma \;\; \xi\eta \;\; \eta\varsigma \;\; \varsigma\xi \;\; \xi\eta\varsigma]\{\alpha\}$

$\mathrm{u} = [1 \;\; \xi \;\; \eta \;\; \varsigma \;\; \xi\eta \;\; \eta\varsigma \;\; \varsigma\xi \;\; \xi\eta\varsigma][A]^{-1}\{u\}_e$

$u = [N]\{u\}_e$

where

$[\mathrm{N}] = [1 \;\; \xi \;\; \eta \;\; \varsigma \;\; \xi\eta \;\; \eta\varsigma \;\; \varsigma\xi \;\; \xi\eta\varsigma][A]^{-1}$

$$[\mathrm{N}] = [1 \;\; \xi \;\; \eta \;\; \varsigma \;\; \xi\eta \;\; \eta\varsigma \;\; \varsigma\xi \;\; \xi\eta\varsigma]_{1X8} \cdot \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{pmatrix}_{8X8}$$

$$[N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_8] =$$

$$\Big(\frac{1}{8}-\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}-\frac{\eta}{8}+\frac{\eta\varsigma}{8}+\frac{\xi}{8}-\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8}$$

$$\frac{1}{8}-\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}+\frac{\eta}{8}-\frac{\eta\varsigma}{8}+\frac{\xi}{8}+\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8}$$

$$\frac{1}{8}+\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}+\frac{\eta}{8}+\frac{\eta\varsigma}{8}+\frac{\xi}{8}+\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8}$$

$$\frac{1}{8}+\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}-\frac{\eta}{8}-\frac{\eta\varsigma}{8}+\frac{\xi}{8}-\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8}$$

$$\frac{1}{8}-\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}-\frac{\eta}{8}+\frac{\eta\varsigma}{8}-\frac{\xi}{8}+\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8}$$

$$\frac{1}{8}-\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}+\frac{\eta}{8}-\frac{\eta\varsigma}{8}-\frac{\xi}{8}-\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8}$$

$$\frac{1}{8}+\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}+\frac{\eta}{8}+\frac{\eta\varsigma}{8}-\frac{\xi}{8}-\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8}$$

$$\frac{1}{8}+\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}-\frac{\eta}{8}-\frac{\eta\varsigma}{8}-\frac{\xi}{8}+\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8}\Big)$$

Equating we get

$$N_1=\frac{1}{8}-\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}-\frac{\eta}{8}+\frac{\eta\varsigma}{8}+\frac{\xi}{8}-\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8} \quad (5)$$

$$N_2=\frac{1}{8}-\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}+\frac{\eta}{8}-\frac{\eta\varsigma}{8}+\frac{\xi}{8}+\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8} \quad (6)$$

$$N_3=\frac{1}{8}+\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}+\frac{\eta}{8}+\frac{\eta\varsigma}{8}+\frac{\xi}{8}+\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8} \quad (7)$$

$$N_4=\frac{1}{8}+\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}-\frac{\eta}{8}-\frac{\eta\varsigma}{8}+\frac{\xi}{8}-\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8} \quad (8)$$

$$N_5=\frac{1}{8}-\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}-\frac{\eta}{8}+\frac{\eta\varsigma}{8}-\frac{\xi}{8}+\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8} \quad (9)$$

$$N_6=\frac{1}{8}-\frac{\varsigma}{8}+\frac{\varsigma\xi}{8}+\frac{\eta}{8}-\frac{\eta\varsigma}{8}-\frac{\xi}{8}-\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8} \quad (10)$$

$$N_7=\frac{1}{8}+\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}+\frac{\eta}{8}+\frac{\eta\varsigma}{8}-\frac{\xi}{8}-\frac{\xi\eta}{8}-\frac{\xi\eta\varsigma}{8} \quad (11)$$

$$N_8=\frac{1}{8}+\frac{\varsigma}{8}-\frac{\varsigma\xi}{8}-\frac{\eta}{8}-\frac{\eta\varsigma}{8}-\frac{\xi}{8}+\frac{\xi\eta}{8}+\frac{\xi\eta\varsigma}{8} \quad (12)$$

$N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8$ are called shape functions

IV. VERIFICATION

(I) 1st Conditon

Sum of all the shape functions is equal to one

$N_1+N_2+N_3+N_4+N_5+N_6+N_7+N_8$ = (5)+(6)+(7)+(8)+(9)+(10)+(11)+(12)

Output

$N_1+N_2+N_3+N_4+N_5+N_6+N_7+N_8=1$

(II) 2nd Condition

Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 (1,-1,-1) ξ:=1, η:=-1, ς:=-1	*At* Node 2 (1,1,-1) ξ:=1, η:=1, ς:=-1
N_1	N_1
N_2	N_2
N_3	N_3
N_4	N_4
N_5	N_5
N_6	N_6
N_7	N_7
N_8	N_8
Output	Output
1	0
0	1
0	0
0	0
0	0
0	0
0	0
0	0

At Node 3 (1,1,1)	*At* Node 4 (1,-1,1)
$\xi:=1,\ \eta:=1, \varsigma:=1$	$\xi:=1,\ \eta:=-1, \varsigma:=1$
N_1	N_1
N_2	N_2
N_3	N_3
N_4	N_4
N_5	N_5
N_6	N_6
N_7	N_7
N_8	N_8
Output	Output
0	0
0	0
1	0
0	1
0	0
0	0
0	0
0	0

At Node 5 (-1,-1,-1)	*At* Node 6 (-1,1,-1)
$\xi:=-1,\ \eta:=-1, \varsigma:=-1$	$\xi:=-1,\ \eta:=1, \varsigma:=-1$
N_1	N_1
N_2	N_2
N_3	N_3
N_4	N_4
N_5	N_5
N_6	N_6
N_7	N_7
N_8	N_8

Output	Output
0	0
0	0
0	0
0	0
1	0
0	1
0	0
0	0

At Node 7 (-1,1,1)	*At* Node 8 (-1,-1,1)
$\xi:=-1,\ \eta:=1, \varsigma:=1$	$\xi:=-1,\ \eta:=-1, \varsigma:=1$
N_1	N_1
N_2	N_2
N_3	N_3
N_4	N_4
N_5	N_5
N_6	N_6
N_7	N_7
N_8	N_8
Output	Output
0	0
0	0
0	0
0	0
0	0
0	0
1	0
0	1

V. CONCLUSIONS

1. Derived Shape functions for hexahedral element.
2. Verified sum of all the shape functions is equal to one.
3. Verified each shape function has a value of one at its own node and zero at the other nodes.

References

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[2]. Mathematica 9 Software, Wolfram Research, Version number 9.0.0.0, 1988-2012.