Deriving Shape Functions for Hexahedron Element by Lagrange Functions and Verified

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***Abstract* —** *In this paper, I derived shape functions for hexahedron element by lagrange functions and also I verified two verification conditions for shape functions. First verification condition is sum of all the shape functions is equal to one and second verification condition is each shape function has a value of one at its own node and zero at the other nodes. For computational purpose I used Mathematica 9 Software [2].*

Keywords - *Hexahedron element, Lagrange functions, Shape functions.*

I.INTRODUCTION

Interpolation functions derived using the dependent unknown-not its derivatives-at the nodes i.e., interpolation functions with C^0 continuity are called the Lagrange family of interpolation functions or Shape functions[3].

II. GEOMETRICAL DESCREPTION

Hexahedron element with eight nodes is shown in figure.1 with axis ξ,η,ς respectively.

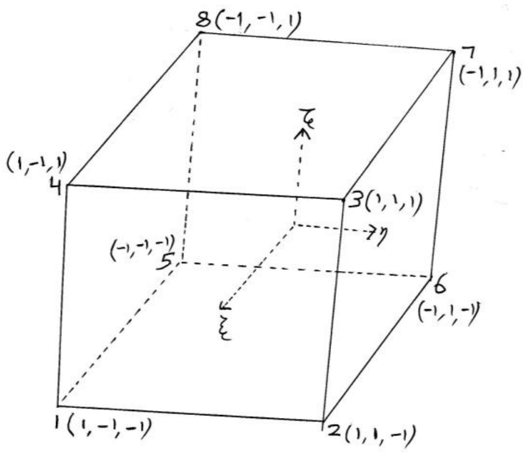

Figure.1 Typical hexahedron element

III. DERIVING SHAPE FUNCTIONS FOR HEXAHEDRON ELEMENT

The typical element is shown in Figure.1

The coordinates for various nodal points are

1(1,-1,-1), 2(1,1,-1), 3(1,1,1), 4(1,-1,1), 5(-1,-1,-1),6(-1,1,-1), 7(-1,1,1), 8(-1,-1,1)

For the C^0 continuity element in three dimensions

$$N_i = L_i(\xi)L_i(\eta)L_i(\varsigma) \qquad (1)$$

where L_i refers to Lagrangian function at node i.

In Figure.1 there are 3 nodes in each direction. Hence n=3 in Lagrange function.

Lagrange polynomial in three dimension is defined by

$$N_i(\xi,\eta,\varsigma) = \frac{\left(\xi-\xi_{For\ \text{Node i, }\xi\text{-axis node}}\right)}{\left(\xi_i-\xi_{For\ \text{Node i, }\xi\text{-axis node}}\right)}.\frac{\left(\eta-\eta_{For\ \text{Node i, }\eta\text{-axis node}}\right)}{\left(\eta_i-\eta_{For\ \text{Node i, }\eta\text{-axis node}}\right)}.\frac{\left(\varsigma-\varsigma_{For\ \text{Node i, }\varsigma\text{-axis node}}\right)}{\left(\varsigma_i-\varsigma_{For\ \text{Node i, }\varsigma\text{-axis node}}\right)} \qquad (2)$$

ξ_1,η_1,ς_1

For Node 1(1,-1,-1)

i=1

ξ_5,η_5,ς_5

$\xi-axis$ Node 5(-1,-1,-1) and

ξ_2,η_2,ς_2

$\eta-axis$ Node 2(1,1,-1) and

ξ_4,η_4,ς_4

$\varsigma-axis$ Node 4(1,-1,1)

$$(2) \Rightarrow N_1(\xi,\eta,\varsigma) = \frac{(\xi-\xi_5)}{(\xi_1-\xi_5)}\cdot\frac{(\eta-\eta_2)}{(\eta_1-\eta_2)}\cdot\frac{(\varsigma-\varsigma_4)}{(\varsigma_1-\varsigma_4)} \quad (3)$$

$$N_1(1,-1,-1) = \frac{(\xi-(-1))}{(1-(-1))}\cdot\frac{(\eta-1)}{(-1-1)}\cdot\frac{(\varsigma-1)}{(-1-1)} \Rightarrow N_1 = \frac{(\xi+1)}{(1+1)}\cdot\frac{(\eta-1)}{(-2)}\cdot\frac{(\varsigma-1)}{(-2)}$$

$$N_1 = \frac{(\xi+1)(\eta-1)(\varsigma-1)}{8} \quad (4)$$

For Node 2(1,1,-1) (ξ_2,η_2,ς_2)

i=2

$\xi-axis$ Node 6(-1,1,-1) (ξ_6,η_6,ς_6) and

$\eta-axis$ Node 1(1,-1,-1) (ξ_1,η_1,ς_1) and

$\varsigma-axis$ Node 3(1,1,1) (ξ_3,η_3,ς_3)

$$(2) \Rightarrow N_1(\xi,\eta,\varsigma) = \frac{(\xi-\xi_6)}{(\xi_2-\xi_6)}\cdot\frac{(\eta-\eta_1)}{(\eta_2-\eta_1)}\cdot\frac{(\varsigma-\varsigma_3)}{(\varsigma_2-\varsigma_3)}$$

$$N_2(1,1,-1) = \frac{(\xi-(-1))}{(1-(-1))}\cdot\frac{(\eta-(-1))}{(1-(-1))}\cdot\frac{(\varsigma-1)}{(-1-1)} \Rightarrow N_2 = \frac{(\xi+1)}{(1+1)}\cdot\frac{(\eta+1)}{1+1}\cdot\frac{(\varsigma-1)}{(-2)}$$

$$N_2 = \frac{(\xi+1)(\eta+1)(\varsigma-1)}{-8} \quad (5)$$

For Node 3(1,1,1) (ξ_3,η_3,ς_3)

i=3

$\xi-axis$ Node 7(-1,1,1) (ξ_7,η_7,ς_7) and

$\eta-axis$ Node 4(1,-1,1) (ξ_4,η_4,ς_4) and

$\varsigma-axis$ Node 2(1,1,-1) (ξ_2,η_2,ς_2)

$$(2) \Rightarrow N_3(\xi,\eta,\varsigma) = \frac{(\xi-\xi_7)}{(\xi_3-\xi_7)}\cdot\frac{(\eta-\eta_4)}{(\eta_3-\eta_4)}\cdot\frac{(\varsigma-\varsigma_2)}{(\varsigma_3-\varsigma_2)}$$

$$N_3(1,1,1) = \frac{(\xi-(-1))}{(1-(-1))}\cdot\frac{(\eta-(-1))}{(1-(-1))}\cdot\frac{(\varsigma-(-1))}{(1-(-1))} \Rightarrow N_3 = \frac{(\xi+1)}{(1+1)}\cdot\frac{(\eta+1)}{(1+1)}\cdot\frac{(\varsigma+1)}{(1+1)}$$

$$N_3 = \frac{(\xi+1)(\eta+1)(\varsigma+1)}{8} \quad (6)$$

For Node 4(1,-1,1) (ξ_4,η_4,ς_4)

i=4

$\xi-axis$ Node 8(-1,-1,1) (ξ_8,η_8,ς_8) and

$\eta-axis$ Node 3(1,1,1) (ξ_3,η_3,ς_3) and

$\varsigma-axis$ Node 1(1,-1,-1) (ξ_1,η_1,ς_1)

$$(2)\Rightarrow N_4(\xi,\eta,\varsigma)=\frac{(\xi-\xi_8)}{(\xi_4-\xi_8)}.\frac{(\eta-\eta_3)}{(\eta_4-\eta_3)}.\frac{(\varsigma-\varsigma_1)}{(\varsigma_4-\varsigma_1)}$$

$$N_4(1,-1,-1)=\frac{(\xi-(-1))}{(1-(-1))}.\frac{(\eta-1)}{(-1-1)}.\frac{(\varsigma-(-1))}{(1-(-1))}\Rightarrow N_4=\frac{(\xi+1)}{(1+1)}.\frac{(\eta-1)}{(-2)}.\frac{(\varsigma+1)}{(1+1)}$$

$$N_4=\frac{(\xi+1)(\eta-1)(\varsigma+1)}{-8} \qquad (7)$$

For Node 5(-1,-1,-1) (ξ_5,η_5,ς_5)

i=5

$\xi-axis$ Node 1(1,-1,-1) (ξ_1,η_1,ς_1) and

$\eta-axis$ Node 6(-1,1,-1) (ξ_6,η_6,ς_6) and

$\varsigma-axis$ Node 8(-1,-1,1) (ξ_8,η_8,ς_8)

$$(2)\Rightarrow N_5(\xi,\eta,\varsigma)=\frac{(\xi-\xi_1)}{(\xi_5-\xi_1)}.\frac{(\eta-\eta_6)}{(\eta_5-\eta_6)}.\frac{(\varsigma-\varsigma_8)}{(\varsigma_5-\varsigma_8)}$$

$$N_5(-1,-1,-1)=\frac{(\xi-1)}{(-1-1)}.\frac{(\eta-1)}{(-1-1)}.\frac{(\varsigma-1)}{(-1-1)}\Rightarrow N_5=\frac{(\xi-1)}{-2}.\frac{(\eta-1)}{-2}.\frac{(\varsigma-1)}{-2}$$

$$N_5=\frac{(\xi-1)(\eta-1)(\varsigma-1)}{-8} \qquad (8)$$

For Node 6(-1,1,-1) (ξ_6,η_6,ς_6)

i=6

$\xi-axis$ Node 2(1,1,-1) (ξ_2,η_2,ς_2) and

$\eta-axis$ Node 5(-1,-1,-1) (ξ_5,η_5,ς_5) and

$\varsigma-axis$ Node 7(-1,1,1) (ξ_7,η_7,ς_7)

$$(2)\Rightarrow N_6(\xi,\eta,\varsigma)=\frac{(\xi-\xi_2)}{(\xi_6-\xi_2)}.\frac{(\eta-\eta_5)}{(\eta_6-\eta_5)}.\frac{(\varsigma-\varsigma_7)}{(\varsigma_6-\varsigma_7)}$$

$$N_6(-1,-1,1)=\frac{(\xi-1)}{(-1-1)}.\frac{(\eta-(-1))}{(1-(-1))}.\frac{(\varsigma-1)}{(-1-1)}\Rightarrow N_6=\frac{(\xi-1)}{-2}.\frac{(\eta+1)}{2}.\frac{(\varsigma-1)}{-2}$$

$$N_6=\frac{(\xi-1)(\eta+1)(\varsigma-1)}{8} \qquad (9)$$

For Node 7(-1,1,1) (ξ_7,η_7,ς_7)

i=7

$\xi-axis$ Node 3(1,1,1) (ξ_3,η_3,ς_3) and

$\eta-axis$ Node 8(-1,-1,1) (ξ_8,η_8,ς_8) and

$\varsigma-axis$ Node 6(-1,1,-1) (ξ_6,η_6,ς_6)

$$(2) \Rightarrow N_7(\xi,\eta,\varsigma) = \frac{(\xi-\xi_3)}{(\xi_7-\xi_3)}.\frac{(\eta-\eta_8)}{(\eta_7-\eta_8)}.\frac{(\varsigma-\varsigma_6)}{(\varsigma_8-\varsigma_6)}$$

$$N_7(-1,1,-1) = \frac{(\xi-1)}{(-1-1)}.\frac{(\eta-(-1))}{(1-(-1))}.\frac{(\varsigma-(-1))}{(1-(-1))} \Rightarrow N_6 = \frac{(\xi-1)}{-2}.\frac{(\eta+1)}{2}.\frac{(\varsigma-1)}{2}$$

$$N_7 = \frac{(\xi-1)(\eta+1)(\varsigma+1)}{-8} \qquad (10)$$

For Node 8(-1,-1,1) $\xi_8, \eta_8, \varsigma_8$

i=8

$\xi-axis$ Node 4(1,-1,1) $\xi_4, \eta_4, \varsigma_4$ and

$\eta-axis$ Node 7(-1,1,1) $\xi_7, \eta_7, \varsigma_7$ and

$\varsigma-axis$ Node 5(-1,-1,-1) $\xi_5, \eta_5, \varsigma_5$

$$(2) \Rightarrow N_8(\xi,\eta,\varsigma) = \frac{(\xi-\xi_4)}{(\xi_8-\xi_4)}.\frac{(\eta-\eta_7)}{(\eta_8-\eta_7)}.\frac{(\varsigma-\varsigma_5)}{(\varsigma_8-\varsigma_5)}$$

$$N_8(-1,1,1) = \frac{(\xi-1)}{(-1-1)}.\frac{(\eta-1)}{(-1-1)}.\frac{(\varsigma-(-1))}{(1-(-1))} \Rightarrow N_8 = \frac{(\xi-1)}{-2}.\frac{(\eta-1)}{-2}.\frac{(\varsigma+1)}{2}$$

$$N_8 = \frac{(\xi-1)(\eta-1)(\varsigma+1)}{8} \qquad (11)$$

N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8 are called shape functions

IV. VERIFICATION

(I) 1st Conditon

Sum of all the shape functions is equal to one

$N_1+N_2+N_3+N_4+N_5+N_6+N_7+N_8$ = (5)+(6)+(7)+(8)+(9)+(10)+(11)+(12)

Output

$N_1+N_2+N_3+N_4+N_5+N_6+N_7+N_8=1$

(II) 2^{nd} Condition

Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 (1,-1,-1)	*At* Node 2 (1,1,-1)
ξ:=1, η:=-1,ς:=-1	ξ:=1, η:=1,ς:=-1
N_1	N_1
N_2	N_2
N_3	N_3
N_4	N_4
N_5	N_5
N_6	N_6
N_7	N_7
N_8	N_8
Output	Output
1	0
0	1
0	0
0	0
0	0
0	0
0	0
0	0

At Node 3 (1,1,1)	*At* Node 4 (1,-1,1)
ξ:=1, η:=1,ς:=1	ξ:=1, η:=-1,ς:=1
N_1	N_1
N_2	N_2
N_3	N_3
N_4	N_4
N_5	N_5
N_6	N_6

N_7 N_8 Output 0 0 1 0 0 0 0 0	N_7 N_8 Output 0 0 0 1 0 0 0 0

At Node 5 (-1,-1,-1)	*At* Node 6 (-1,1,-1)
ξ:=-1, η:=-1,ς:=-1 N_1 N_2 N_3 N_4 N_5 N_6 N_7 N_8 Output 0 0 0 0 1 0 0 0	ξ:=-1, η:=1,ς:=-1 N_1 N_2 N_3 N_4 N_5 N_6 N_7 N_8 Output 0 0 0 0 0 1 0 0

At Node 7 (-1,1,1)	*At* Node 8 (-1,-1,1)
ξ:=-1, η:=1,ς:=1 N_1 N_2 N_3 N_4 N_5 N_6 N_7 N_8 Output 0 0 0 0 0 0 1 0	ξ:=-1, η:=-1,ς:=1 N_1 N_2 N_3 N_4 N_5 N_6 N_7 N_8 Output 0 0 0 0 0 0 0 1

V. CONCLUSIONS

1. Derived Shape functions for hexahedron element.
2. Verified sum of all the shape functions is equal to one
3. Verified each shape function has a value of one at its own node and zero at the other nodes.

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