Deriving Shape Functions for 2,3,4,5 Noded Line Element by Lagrange Functions and Verified

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***Abstract* —** In *this paper, I derived shape functions for 2,3,4,5 noded line element by lagrange functions and also I verified two verification conditions for shape functions. First verification condition is sum of all the shape functions is equal to one and second verification condition is each shape function has a value of one at its own node and zero at the other nodes. For computational purpose I used Mathematica 9 Software [2].*

Keywords — *Line element, Lagrange functions, Shape functions.*

I. INTRODUCTION

Lagrange polynomials can be straight way used as shape functions for one dimensional problems. Line element with 2,3,4,5,6,7,8,9,10 nodes is shown in figures.1,2,3,4,5,6,7,8,9. In finite element method, the complex domain like a complicated structure is discretized into many number of finite elements and the solution like nodal displacements, element stresses are evaluated in the various locations of the structure. For achieving the nodal displacements in all parts of the element, we have to depend on mathematical relation namely shape functions.

II. GEOMETRICAL DESCREPTION

Line element with 2,3,4,5 nodes is shown in figures.1,2,3,4.

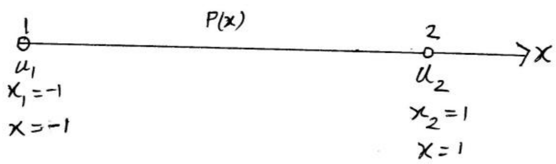

Figure.1: 2 noded line element

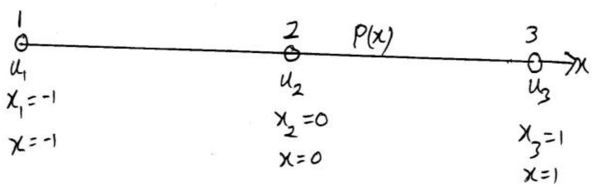

Figure.2: 3 noded line element

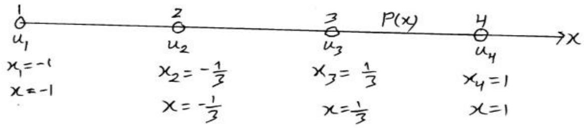

Figure.3: 4 noded line element

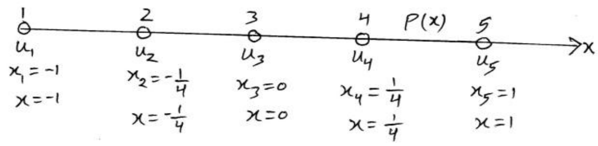

Figure.4: 5 noded line element

III. DERIVING SHAPE FUNCTIONS FOR 2,3,4,5 NODED LINE ELEMENT

(i) The typical 2 noded element is shown in Figure.1.

Shape function for node 1 is N_1 and for node 2 is N_2

$x_1 := -1$

$x_2 := 1$

Lagrange function for node 1&2

$$N_1 := \frac{(x - x_2)}{(x_1 - x_2)}$$

$$N_2 := \frac{(x - x_1)}{(x_2 - x_1)}$$

N_1

N_2

Output

$\frac{1-x}{2}$

$\frac{1+x}{2}$

$\therefore$ *For* 2 noded element shape functions are

$N_1 = \frac{1-x}{2}$ and $N_2 = \frac{1+x}{2}$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

$N_1 := \frac{1-x}{2}$

$N_2 := \frac{1+x}{2}$

$FullSimplifly[N_1 + N_2]$

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get

$N_1 = 1,\ N_2 = 0$

At Node 2 x = 1 then we get

$N_1 = 0,\ N_2 = 1$

(ii) The typical 3 noded element is shown in Figure.2.

Shape function for node 1 is N_1, for node 2 is N_2 and for node 3 is N_3

$x_1 := -1$

$x_2 := 0$

$x_3 := 1$

Lagrange function for node 1,2&3

$$N_1 := \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$N_2 := \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_3 := \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

N_1

N_2

N_3

Output

$\frac{1}{2}(-1+x)x$

$-(-1+x)(1+x)$

$\frac{1}{2}x(1+x)$

$\therefore$ *For* 3 noded element shape functions are

$N_1 = \frac{1}{2}(-1+x)x$, $N_2 = -(-1+x)(1+x)$

and $N_3 = \frac{1}{2}x(1+x)$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

$N_1 := \frac{1}{2}(-1+x)x \quad N_2 := -(-1+x)(1+x)$

$N_3 := \frac{1}{2}x(1+x)$

FullSimplify[$N_1 + N_2 + N_3$]

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get $N_1 = 1$, $N_2 = 0, N_3 = 0$

At Node 2 x = 0 then we get $N_1 = 0$, $N_2 = 1, N_3 = 0$

At Node 3 x = 1 then we get

$N_1 = 0,\ N_2 = 0, N_3 = 1$

(iii) The typical 4 noded element is shown in Figure.3.

Shape function for node 1 is N_1, for node 2 is N_2, for node 3 is N_3 and for node 4 is N_4.

$x_1 := -1$

$x_2 := -\frac{1}{3}$

$x_3 := \frac{1}{3}$

$x_4 := 1$

Lagrange function for node 1,2,3&4

$$N_1 := \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$N_2 := \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$N_3 := \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$N_4 := \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

N_1

N_2

N_3

N_4

Output

$$-\frac{9}{16}(-1+x)\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)$$

$$\frac{27}{16}(-1+x)\left(-\frac{1}{3}+x\right)(1+x)$$

$$-\frac{27}{16}(-1+x)\left(\frac{1}{3}+x\right)(1+x)$$

$$\frac{9}{16}\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)(1+x)$$

$\therefore$ *For* 4 noded element shape functions are

$$\mathrm{N}_1 = -\frac{9}{16}(-1+x)\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right),$$

$$\mathrm{N}_2 = \frac{27}{16}(-1+x)\left(-\frac{1}{3}+x\right)(1+x),$$

$$\mathrm{N}_3 = \frac{27}{16}(-1+x)\left(-\frac{1}{3}+x\right)(1+x) \text{ and}$$

$$\mathrm{N}_4 = \frac{9}{16}\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)(1+x)$$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

$$\mathrm{N}_1 := -\frac{9}{16}(-1+x)\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)$$

$$\mathrm{N}_2 := \frac{27}{16}(-1+x)\left(-\frac{1}{3}+x\right)(1+x)$$

$$\mathrm{N}_3 := \frac{27}{16}(-1+x)\left(-\frac{1}{3}+x\right)(1+x) \text{ and}$$

$$\mathrm{N}_4 := \frac{9}{16}\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)(1+x)$$

FullSimplify[$\mathrm{N}_1 + \mathrm{N}_2 + \mathrm{N}_3 + \mathrm{N}_4$]

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get

$\mathrm{N}_1 = 1,\ \mathrm{N}_2 = 0, \mathrm{N}_3 = 0, N_4 = 0$

At Node 2 x = $-\frac{1}{3}$ then we get

$\mathrm{N}_1 = 0,\ \mathrm{N}_2 = 1, \mathrm{N}_3 = 0, \mathrm{N}_4 = 0$

At Node 3, x = $\frac{1}{3}$ then we get

$\mathrm{N}_1 = 0,\ \mathrm{N}_2 = 0, \mathrm{N}_3 = 1, \mathrm{N}_4 = 0$

At Node 4, x = 1 then we get

$\mathrm{N}_1 = 0,\ \mathrm{N}_2 = 0, \mathrm{N}_3 = 0, \mathrm{N}_4 = 1$

(iv) The typical 5 noded element is shown in Figure.4.

Shape function for node 1 is N_1, for node 2 is N_2, for node 3 is N_3, for node 4 is N_4 and for node 5 is N_5.

$x_1 := -1$

$x_2 := -\frac{1}{4}$

$x_3 := 0$

$x_4 := \frac{1}{4}$

$x_5 := 1$

Lagrange function for node 1,2,3,4&5.

$$N_1 := \frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}$$
$$N_2 := \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}$$
$$N_3 := \frac{(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}$$
$$N_4 := \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}$$
$$N_5 := \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}$$

N_1

N_2

N_3

N_4

N_5

Output

$$\frac{8}{15}(-1+x)\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)$$
$$-\frac{128}{15}(-1+x)\left(-\frac{1}{4}+x\right)x(1+x)$$
$$16(-1+x)\left(-\frac{1}{4}+x\right)\left(\frac{1}{4}+x\right)(1+x)$$
$$-\frac{128}{15}(-1+x)x\left(\frac{1}{4}+x\right)(1+x)$$
$$\frac{8}{15}\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)(1+x)$$

$\therefore$ *For* 5 noded element shape functions are

$$\mathrm{N}_1 = \frac{8}{15}(-1+x)\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right),$$
$$\mathrm{N}_2 = -\frac{128}{15}(-1+x)\left(-\frac{1}{4}+x\right)x(1+x)\ ,$$
$$\mathrm{N}_3 = 16(-1+x)\left(-\frac{1}{4}+x\right)\left(\frac{1}{4}+x\right)(1+x)\ ,$$
$$\mathrm{N}_4 = -\frac{128}{15}(-1+x)x\left(\frac{1}{4}+x\right)(1+x)$$
$$\text{and } \mathrm{N}_5 = \frac{8}{15}\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)(1+x)$$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

$$\mathrm{N}_1 := \frac{8}{15}(-1+x)\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right),$$
$$\mathrm{N}_2 := -\frac{128}{15}(-1+x)\left(-\frac{1}{4}+x\right)x(1+x)\ ,$$
$$\mathrm{N}_3 := 16(-1+x)\left(-\frac{1}{4}+x\right)\left(\frac{1}{4}+x\right)(1+x)\ ,$$
$$\mathrm{N}_4 := -\frac{128}{15}(-1+x)x\left(\frac{1}{4}+x\right)(1+x)$$
$$\text{and } \mathrm{N}_5 := \frac{8}{15}\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)(1+x)$$

FullSimplify[$\mathrm{N}_1 + \mathrm{N}_2 + \mathrm{N}_3 + \mathrm{N}_4 + \mathrm{N}_5$]

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get

$\mathrm{N}_1 = 1,\ \mathrm{N}_2 = 0, \mathrm{N}_3 = 0, N_4 = 0,$
$N_5 = 0$

At Node 2 x = $-\frac{1}{4}$ then we get

$\mathrm{N}_1 = 0,\ \mathrm{N}_2 = 1, \mathrm{N}_3 = 0, \mathrm{N}_4 = 0,$
$\mathrm{N}_5 = 0$

At Node 3, x = 0 then we get

$\mathrm{N}_1 = 0,\ \mathrm{N}_2 = 0, \mathrm{N}_3 = 1, \mathrm{N}_4 = 0,$
$\mathrm{N}_5 = 0$

At Node 4, x = $\frac{1}{4}$ then we get

$\mathrm{N}_1 = 0,\ \mathrm{N}_2 = 0, \mathrm{N}_3 = 0, \mathrm{N}_4 = 1,$
$\mathrm{N}_5 = 0$

At Node 5, x = 1 then we get

$\mathrm{N}_1 = 0,\ \mathrm{N}_2 = 0, \mathrm{N}_3 = 0, \mathrm{N}_4 = 0,$
$\mathrm{N}_5 = 1$

IV. CONCLUSIONS

1. Derived Shape functions for 2,3,4,5 noded line element by lagrange functions.

2. Verified sum of all the shape functions is equal to one
3. Verified each shape function has a value of one at its own node and zero at the other nodes.

REFERENCES

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