Deriving Shape Functions for 9-Noded Rectangular Element by using Lagrange Functions in Natural Coordinate System and Verified

P. Reddaiah[#1]

[#] Professor of Mathematics, Global College of Engineering and Technology, kadapa, Andhra Pradesh, India.

Abstract* — In this paper, I derived shape functions for 9-noded rectangular element by using Lagrange functions in natural coordinate system and also I verified two verification conditions for shape functions. First verification condition is sum of all the shape functions is equal to one and second verification condition is each shape function has a value of one at its own node and zero at the other nodes. For computational purpose I used Mathematica 9 Software [2].

Keywords — *Rectangualar element, Lagrange functions, Shape functions.*

1. INTRODUCTION

In engineering problems there are some basic unknowns. If they are found, the behaviour of the entire structure can be predicted. The basic unknowns or the field variables which are encountered in the engineering problems are displacements in solid mechanics, velocities in fluid mechanics, electric and magnetic potentials in electrical engineering and temperatures in heat flow problems.

In a continuum, these unknowns are infinite. The finite element procedure reduces such unknowns to a finite number by dividing the solution region into small parts called elements and by expressing the unknown field variables interms of assumed approximating functions (Interpolating functions/Shape functions) within each element. The approximating functions are defined in terms of field variables of specified points called nodes or nodal points. Thus in the finite element analysis the unknowns are the field variables of the nodal points. Once these are found the field variables at any point can be found by using interpolation functions/Shape functions.

II. GEOMETRICAL DESCREPTION

Typical nine noded element is shown in Fig1.1

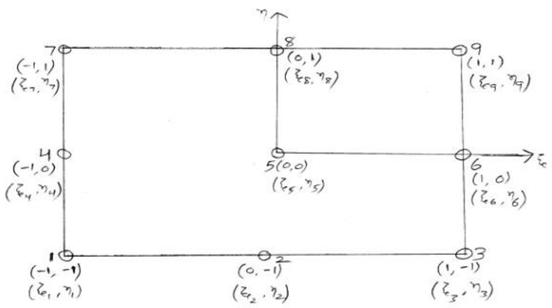

Fig.1 Typical nine noded rectangular element.

III. DERIVING SHAPE FUNCTIONS FOR NINE NODED RECTANGULAR ELEMENT BY USING LAGRANGE FUNCTIONS.

The natural coordinates of various nodes are as shown in the Fig.1.

Nodal unknowns

Basic unknowns may be displacements for stress analysis, temperatures for heat flow problems and the potentials for fluid flow or in the magnetic field problems. In the problems like truss analysis, plane stress and plane strain, it is enough if the continuity of only displacements are satisfied, since there is no change in the slopes at any nodal point. Such problems are classified as 'Zeroth' Continuity problems and are indicated as C^0 – Continutiy problem.

For the C^0 Continutiy element in two dimensions

$$N_i = L_i(\xi)L_i(\eta) \qquad (1)$$

Where L_i refers to Lagrangian function at node i.

In Fig.1 there are 3 nodes in each direction.
Hence n=3 in Lagrange function.
Lagrange Polynomial in one dimension is defined by

$$\mathrm{L_k}(x)=\prod_{\substack{m=1\\ m\neq k}}^{n}\frac{x-x_m}{x_k-x_m}\ ,\ \mathrm{L_k}(y)=\prod_{\substack{m=1\\ m\neq k}}^{n}\frac{y-y_m}{y_k-y_m}$$

When i=1
n=3, k=1, x=$\xi, y=\eta$

$(1)\Rightarrow N_1=L_1(\xi)L_1(\eta)$
At node 1, *Along* ξ-axis nodes 1,2,3
$(\textit{For } L_1(\xi)$ we should take nodes 2 & 3$)$
At node 1, *Along* η-axis nodes 1,4,7
$(\textit{For } L_1(\eta)$ we should take nodes 4 & 7$)$

$$N_1=\frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)}\frac{(\eta-\eta_4)(\eta-\eta_7)}{(\eta_1-\eta_4)(\eta_1-\eta_7)}$$

$$N_1=\frac{(\xi-0)(\xi-1)}{(-1-0)(-1-1)}\frac{(\eta-0)(\eta-1)}{(-1-0)(-1-1)}$$

$$N_1=\frac{\xi(\xi-1)}{-1(-2)}\frac{\eta(\eta-1)}{-1(-2)}$$

$$N_1=\frac{\xi(\xi-1)}{2}\frac{\eta(\eta-1)}{2}$$

$$N_1=\frac{\xi(\xi-1)\eta(\eta-1)}{4} \qquad (2)$$

When i=2
n=3, k=2, x=$\xi, y=\eta$
$(1)\Rightarrow N_2=L_2(\xi)L_2(\eta)$
At node 2, *Along* ξ-axis nodes 2,1,3
$(\textit{For } L_2(\xi)$ we should take nodes 1 & 3$)$
At node 2, *Along* η-axis nodes 2,5,8
$(\textit{For } L_2(\eta)$ we should take nodes 5 & 8$)$

$$N_2=\frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}\frac{(\eta-\eta_5)(\eta-\eta_8)}{(\eta_2-\eta_5)(\eta_2-\eta_8)}$$

$$N_2=\frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)}\frac{(\eta-0)(\eta-1)}{(-1-0)(-1-1)}$$

$$N_2=\frac{(\xi+1)(\xi-1)}{(0+1)(-1)}\frac{\eta(\eta-1)}{(-1)(-2)}$$

$$N_2=\frac{(\xi+1)(\xi-1)}{(1)(-1)}\frac{\eta(\eta-1)}{2}$$

$$N_2=\frac{(\xi+1)(\xi-1)}{-1}\frac{\eta(\eta-1)}{2}$$

$$N_2=\frac{(\xi+1)(\xi-1)\eta(\eta-1)}{-2} \qquad (3)$$

When i=3
n=3, k=3, x=$\xi, y=\eta$
$(1)\Rightarrow N_3=L_3(\xi)L_3(\eta)$
At node 3, *Along* ξ-axis nodes 3,1,2
$(\textit{For } L_3(\xi)$ we should take nodes 1 & 2$)$
At node 3, *Along* η-axis nodes 3,6,9
$(\textit{For } L_3(\eta)$ we should take nodes 6 & 9$)$

$$N_3=\frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}\frac{(\eta-\eta_6)(\eta-\eta_9)}{(\eta_3-\eta_6)(\eta_3-\eta_9)}$$

$$N_3=\frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)}\frac{(\eta-0)(\eta-1)}{(-1-0)(-1-1)}$$

$$N_3=\frac{(\xi+1)\xi}{(1+1)(1)}\frac{\eta(\eta-1)}{(-1)(-2)}$$

$$N_3=\frac{(\xi+1)\xi}{(2)(1)}\frac{\eta(\eta-1)}{2}$$

$$N_3=\frac{(\xi+1)\xi\eta(\eta-1)}{4} \qquad (4)$$

When i=4
n=3, k=4, x=$\xi, y=\eta$
$(1)\Rightarrow N_4=L_4(\xi)L_4(\eta)$
At node 4, *Along* ξ-axis nodes 4,5,6
$(\textit{For } L_4(\xi)$ we should take nodes 5 & 6$)$
At node 4, *Along* η-axis nodes 4,1,7
$(\textit{For } L_4(\eta)$ we should take nodes 1 & 7$)$

$$N_4=\frac{(\xi-\xi_5)(\xi-\xi_6)}{(\xi_4-\xi_5)(\xi_4-\xi_6)}\frac{(\eta-\eta_1)(\eta-\eta_7)}{(\eta_4-\eta_1)(\eta_4-\eta_7)}$$

$$N_4=\frac{(\xi-(0))(\xi-1)}{(-1-(0))(-1-(-1))}\frac{(\eta-(-1))(\eta-1)}{(0-(-1))(0-1)}$$

$$N_4=\frac{\xi(\xi-1)}{(-1)(-1-1)}\frac{(\eta+1)(\eta-1)}{(0+1)(0-1)}$$

$$N_4=\frac{\xi(\xi-1)}{(-1)(-2)}\frac{(\eta+1)(\eta-1)}{(1)(-1)}$$

$$N_3=\frac{\xi(\xi-1)(\eta+1)(\eta-1)}{-2} \qquad (5)$$

When i=5
n=3, k=5, x=$\xi, y=\eta$

$(1) \Rightarrow N_5 = L_5(\xi)L_5(\eta)$

At node 5, *Along* ξ-axis nodes 5,4,6

$(For\ L_5(\xi)$ we should take nodes 4 & 6$)$

At node 5, *Along* η-axis nodes 5,2,8

$(For\ L_5(\eta)$ we should take nodes 2 & 8$)$

$$N_5 = \frac{(\xi-\xi_4)(\xi-\xi_6)}{(\xi_5-\xi_4)(\xi_5-\xi_6)}\frac{(\eta-\eta_2)(\eta-\eta_8)}{(\eta_5-\eta_2)(\eta_5-\eta_8)}$$

$$N_5 = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)}\frac{(\eta-(-1))(\eta-1)}{(0-(-1))(0-1)}$$

$$N_5 = \frac{(\xi+1)(\xi-1)}{(0+1)(-1)}\frac{(\eta+1)(\eta-1)}{(0+1)(-1)}$$

$$N_5 = \frac{(\xi+1)(\xi-1)}{(1)(-1)}\frac{(\eta+1)(\eta-1)}{(1)(-1)}$$

$$N_5 = \frac{(\xi+1)(\xi-1)(\eta+1)(\eta-1)}{1} \qquad (6)$$

When i=6

n=3, k=6, x=$\xi, y=\eta$

$(1) \Rightarrow N_6 = L_6(\xi)L_6(\eta)$

At node 6, *Along* ξ-axis nodes 6,4,5

$(For\ L_6(\xi)$ we should take nodes 4 & 5$)$

At node 6, *Along* η-axis nodes 6,3,9

$(For\ L_6(\eta)$ we should take nodes 3 & 9$)$

$$N_6 = \frac{(\xi-\xi_4)(\xi-\xi_5)}{(\xi_6-\xi_4)(\xi_6-\xi_5)}\frac{(\eta-\eta_3)(\eta-\eta_9)}{(\eta_6-\eta_3)(\eta_6-\eta_9)}$$

$$N_6 = \frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)}\frac{(\eta-(-1))(\eta-1)}{(0-(-1))(0-1)}$$

$$N_6 = \frac{(\xi+1)(\xi)}{(1+1)(1)}\frac{(\eta+1)(\eta-1)}{(0+1)(-1)}$$

$$N_6 = \frac{(\xi+1)\xi}{(2)(1)}\frac{(\eta+1)(\eta-1)}{(1)(-1)}$$

$$N_6 = \frac{(\xi+1)\xi(\eta+1)(\eta-1)}{-2} \qquad (7)$$

When i=7

n=3, k=7, x=$\xi, y=\eta$

$(1) \Rightarrow N_7 = L_7(\xi)L_7(\eta)$

At node 7, *Along* ξ-axis nodes 7,8,9

$(For\ L_7(\xi)$ we should take nodes 8 & 9$)$

At node 7, *Along* η-axis nodes 7,4,1

$(For\ L_7(\eta)$ we should take nodes 4 & 1$)$

$$N_7 = \frac{(\xi-\xi_8)(\xi-\xi_9)}{(\xi_7-\xi_8)(\xi_7-\xi_9)}\frac{(\eta-\eta_1)(\eta-\eta_4)}{(\eta_7-\eta_1)(\eta_7-\eta_4)}$$

$$N_7 = \frac{(\xi-0)(\xi-1)}{(-1-0)(-1-1)}\frac{(\eta-(-1))(\eta-0)}{(1-(-1))(1-0)}$$

$$N_7 = \frac{\xi(\xi-1)}{(-1)(-2)}\frac{(\eta+1)\eta}{(1+1)(1)}$$

$$N_7 = \frac{\xi(\xi-1)}{(-1)(-2)}\frac{(\eta+1)\eta}{(1+1)(1)}$$

$$N_7 = \frac{\xi(\xi-1)\xi(\eta+1)\eta}{(-1)(-2)(2)(1)}$$

$$N_7 = \frac{\xi(\xi-1)(\eta+1)\eta}{4} \qquad (8)$$

When i=8

n=3, k=8, x=$\xi, y=\eta$

$(1) \Rightarrow N_8 = L_8(\xi)L_8(\eta)$

At node 8, *Along* ξ-axis nodes 8,7,9

$(For\ L_8(\xi)$ we should take nodes 7 & 9$)$

At node 8, *Along* η-axis nodes 8,5,2

$(For\ L_8(\eta)$ we should take nodes 5 & 2$)$

$$N_8 = \frac{(\xi-\xi_7)(\xi-\xi_9)}{(\xi_8-\xi_7)(\xi_8-\xi_9)}\frac{(\eta-\eta_2)(\eta-\eta_5)}{(\eta_8-\eta_2)(\eta_8-\eta_5)}$$

$$N_8 = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)}\frac{(\eta-(-1))(\eta-0)}{(1-(-1))(1-0)}$$

$$N_8 = \frac{(\xi+1)(\xi-1)}{(0+1)(-1)}\frac{(\eta+1)\eta}{(1+1)(1)}$$

$$N_8 = \frac{(\xi+1)(\xi-1)}{(1)(-1)}\frac{(\eta+1)\eta}{(2)(1)}$$

$$N_7 = \frac{(\xi+1)(\xi-1)(\eta+1)\eta}{-2} \qquad (9)$$

When i=9

n=3, k=9, x=$\xi, y=\eta$

$(1) \Rightarrow N_9 = L_9(\xi)L_9(\eta)$

At node 9, *Along* ξ-axis nodes 9,7,8

$(For\ L_9(\xi)$ we should take nodes 7 & 8$)$

At node 9, *Along* η-axis nodes 9,6,3

$(For\ L_9(\eta)$ we should take nodes 6 & 3$)$

$$N_9 = \frac{(\xi-\xi_7)(\xi-\xi_8)}{(\xi_9-\xi_7)(\xi_9-\xi_8)}\frac{(\eta-\eta_3)(\eta-\eta_6)}{(\eta_9-\eta_3)(\eta_9-\eta_6)}$$

$$N_9 = \frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)}\frac{(\eta-(-1))(\eta-0)}{(1-(-1))(1-0)}$$

$$N_9 = \frac{(\xi+1)\xi}{(1+1)(1)}\frac{(\eta+1)\eta}{(1+1)(1)}$$

$$N_9 = \frac{(\xi+1)\xi}{(2)(1)}\frac{(\eta+1)\eta}{(2)(1)}$$

$$N_9 = \frac{(\xi+1)\xi(\eta+1)\eta}{4} \qquad (10)$$

IV. VERIFICATION

(I a) Verification 1st Point

$$N_1+N_2+N_3+N_4+N_5+N_6+N_7+$$
$$+N_8+N_9 = \frac{\xi(\xi-1)\eta(\eta-1)}{4}$$
$$+\frac{(\xi+1)(\xi-1)\eta(\eta-1)}{-2}+\frac{(\xi+1)\xi\eta(\eta-1)}{4}+$$
$$+\frac{\xi(\xi-1)(\eta+1)(\eta-1)}{-2}+$$
$$+\frac{(\xi+1)(\xi-1)(\eta+1)(\eta-1)}{1}+$$
$$\frac{(\xi+1)(\xi)(\eta+1)(\eta-1)}{-2}++\frac{\xi(\xi-1)(\eta+1)\eta}{4}+$$
$$+\frac{(\xi+1)(\xi-1)(\eta+1)\eta}{-2}+\frac{(\xi+1)\xi(\eta+1)\eta}{4}$$
$$N_1+N_2+N_3+N_4+N_5+N_6+N_7+$$
$$+N_8+N_9 = 1$$

$\therefore$ *Sum* of all the shape functions is equal to one.

$\therefore 1^{st}$ point is verified.

(II a) Verification 2nd Point

(i) *At* Node 1 (-1,-1)

$\xi=-1, \quad \eta=-1$

when $\xi=-1, \ \eta=-1$

$$N_1 = \frac{\xi(\xi-1)\eta(\eta-1)}{4}$$

$$N_1 = \frac{-1(-1-1)(-1)(-1-1)}{4} =$$

$$= \frac{-1(-2)(-1)(-2)}{4} = \frac{4}{4} = 1$$

$$N_1 = 1$$

$$N_2 = \frac{(\xi+1)(\xi-1)\eta(\eta-1)}{-2}$$

$$N_2 = \frac{(-1+1)(-1-1)(-1)(-1-1)}{-2}$$

$$N_2 = 0$$

$$N_3 = \frac{(\xi+1)(\xi)\eta(\eta-1)}{4}$$

$$N_3 = \frac{(-1+1)(-1)(-1)(-1-1)}{4}$$

$$N_3 = 0$$

$$N_4 = \frac{\xi(\xi-1)(\eta+1)(\eta-1)}{-2}$$

$$N_4 = \frac{-1(-1-1)(-1+1)(-1-1)}{-2}$$

$$N_4 = 0$$

$$N_5 = \frac{(\xi+1)(\xi-1)(\eta+1)(\eta-1)}{1}$$

$$N_5 = \frac{(-1+1)(-1-1)(-1+1)(-1-1)}{1}$$

$$N_5 = 0$$

$$N_6 = \frac{(\xi+1)\xi(\eta+1)(\eta-1)}{-2}$$

$$N_6 = \frac{(-1+1)(-1)(-1+1)(-1-1)}{-2}$$

$$N_6 = 0$$

$$N_7 = \frac{\xi(\xi-1)(\eta+1)\eta}{4}$$

$$N_7 = \frac{(-1)(-1-1)(-1+1)(-1)}{4}$$

$$N_7 = 0$$

$$N_8 = \frac{(\xi+1)(\xi-1)(\eta+1)\eta}{-2}$$

$$N_8 = \frac{(-1+1)(-1-1)(-1+1)(-1)}{-2}$$

$$N_8 = 0$$

$\therefore$ *At* Node 1 $N_1=1, \quad N_2=0, \quad N_3=0,$
$N_4=0, \ N_5=0, \ N_6=0, \ N_7=0,$
$N_8=0, \ N_9=0$

$\therefore$ *At* Node 1 2nd condition is verified.

(ii) *At* Node 2 (0,-1) $\quad \xi=0 \quad \eta=-1$

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output	0	1	0	0	0	0	0	0	0

(*iii*) *At* Node 3 (1,-1) $\xi=1, \eta=-1$

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output 0	0	1	0	0	0	0	0	0

(*iv*) *At* Node 4(-1,0) $\xi=-1, \eta=0$

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output 0	0	0	1	0	0	0	0	0

(*v*) *At* Node 5(0,0) $\xi=0, \eta=0$

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output 0	0	0	0	1	0	0	0	0

(*vi*) *At* Node 6(1,0) $\xi=1, \eta=0$

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output 0	0	0	0	0	1	0	0	0

(*vii*) *At* Node 7(-1,1) $\xi=-1, \eta=1$

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output 0	0	0	0	0	0	1	0	0

(*viii*) *At* Node 8(0,1) $\xi=0, \eta=1$

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output 0	0	0	0	0	0	0	1	0

(*ix*) *At* Node 9(1,1) $\xi=1, \eta=1$

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
Output 0	0	0	0	0	0	0	0	1

V. AUTHOR'S CONTRIBUTION

1. Deriving Shape functions for 9 noded Lagrange element in natural coordinate system by Lagrange method.
2. Sum of all the shape functions is equal to one.
3. Each Shape function has a value of one at its own node and zero at the other nodes.

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