Necessary and Sufficient Conditions for the Existence of a Positive Definite Solution for the Matrix Equation $X+A^*X^{-2}A+B^*X^{-2}B=I$

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Abstract – *In this paper we derive necessary and sufficient conditions for the matrix equation $X+A^*X^{-2}A+B^*X^{-2}B=I$ to have a positive definite solution X, where, I is an $n\times n$ identity matrix and A and B are $n\times n$ nonsingular complex matrices. We use these conditions to present some properties on the matrices A and B. Moreover, relations between the solution X and the matrices A and B are proposed.*

Keywords: – *Nonlinear matrix equation, positive definite solution, necessary and sufficient conditions.*

The math subject classification:
<u>*65-XX*</u> *Numerical analysis*
<u>*15-XX*</u> *Linear and multilinear algebra; matrix theory*

1-INTRODUCTION

Consider the nonlinear matrix equation

$$X+A^*X^{-2}A+B^*X^{-2}B=I \tag{1.1}$$

where I is an $n\times n$ identity matrix, A and B are $n\times n$ nonsingular complex matrices , (A^* , B^* stand for the conjugate transpose of A , B respectively). Throughout the paper, we denote by $r(A)$ the spectral radius of A. $\lambda(A)$ represents the eigenvalues of A and the notation $A>B$ $(A\geq B)$ indicates that $A-B$ is positive definite (semidefinite). Several authors [1-17] have studied the existence, the perturbation analysis for some matrix equations, the rate of convergence as well as the necessary and sufficient conditions of the existence of positive definite solutions of similar kinds of nonlinear matrix equations. We organize this paper as follows: First, in section 2, we propose the necessary and sufficient conditions for the existence of a positive definite solution of Eq. (1.1). In section 3, some applications of the obtained results as well as relations between the solution X and the matrices A and B are given.

2- NECESSARY AND SUFFICIENT CONDITIONS

In this section we drive both necessary and sufficient conditions for the existence of a positive definite solution of the nonlinear matrix equation (1.1).

Theorem 2.1

The matrix equation (1.1) has a solution X (symmetric and positive definite) if and only if A and B have the following factorization:

$$A=(M^*M)Z_1 \;,\quad B=(M^*M)Z_2 \tag{2.1}$$

where M is a nonsingular square matrix and the columns of $\begin{pmatrix} M \\ Z_1 \\ Z_2 \end{pmatrix}$ satisfy $M^*M+Z_1^*Z_1+Z_2^*Z_2=I$. In this case we take $X=(M^*M)$ as a solution of the matrix Eq. (1.1).

Proof:

If equation (1.1) has a solution X, then we can write X as $X=(M^*M)$ for some nonsingular matrix M. In particular M can be chosen to be triangular using Cholesky Decomposition in [18, p.141], so we have

$$X+A^*X^{-2}A+B^*X^{-2}B$$
$$=M^*M+A^*\left(M^*M\right)^{-1}\left(M^*M\right)^{-1}A+B^*\left(M^*M\right)^{-1}\left(M^*M\right)^{-1}B$$

then equation (1.1) can be rewritten as

$$M^*M+\left(M^{-1}M^{*-1}A\right)^*\left(M^{-1}M^{*-1}A\right)$$
$$+\left(M^{-1}M^{*-1}B\right)^*\left(M^{-1}M^{*-1}B\right)=I$$

or equivalently

$$\begin{pmatrix} M \\ M^{-1}M^{*-1}A \\ M^{-1}M^{*-1}B \end{pmatrix}^*\begin{pmatrix} M \\ M^{-1}M^{*-1}A \\ M^{-1}M^{*-1}B \end{pmatrix}=I \qquad (2.2)$$

Let $M^{-1}M^{*-1}A=Z_1$ and $M^{-1}M^{*-1}B=Z_2$.

Then $A=(M^*M)Z_1$, $B=(M^*M)Z_2$

and (2.2) means that $M^*M+Z_1^*Z_1+Z_2^*Z_2=I$.

Conversely, suppose that A and B admit the decomposition (2.1). Set $X=(M^*M)$. Then

$$X+A^*X^{-2}A+B^*X^{-2}B$$
$$=M^*M+\left((M^*M)Z_1\right)^*\left(M^*M\right)^{-2}\left((M^*M)Z_1\right)$$
$$+\left((M^*M)Z_2\right)^*\left(M^*M\right)^{-2}\left((M^*M)Z_2\right)$$
$$=M^*M+\left(Z_1^*M^*M\right)\left(M^*M\right)^{-2}\left((M^*M)Z_1\right)$$
$$+\left(Z_2^*M^*M\right)\left(M^*M\right)^{-2}\left((M^*M)Z_2\right)$$
$$=M^*M+Z_1^*Z_1+Z_2^*Z_2=I$$

i.e., X is a solution to the matrix equation (1.1).

Theorem 2.2

The matrix equation (1.1) has a solution if and only if there exist unitary matrices P, U_2 and U_3 and diagonal matrices $\Gamma>0$, $\Sigma\geq 0$ and $\Psi\geq 0$ with $\left(2\Gamma^2+\Sigma^2+\Psi^2\right)=I$ such that:

$$A=2P^*\Gamma^2PU_2\Sigma P\ ,\ B=2P^*\Gamma^2PU_3\Psi P \qquad (2.3)$$

In this case $X=2P^*\Gamma^2P$ is a solution of equation (1.1).

Proof:

Suppose the matrix equation (1.1) has a solution. Then from Theorem 2.1 A and B admit the forms $A=(M^*M)Z_1$, $B=(M^*M)Z_2$. Since the columns of $\begin{pmatrix} M \\ Z_1 \\ Z_2 \end{pmatrix}$ satisfy $M^*M+Z_1^*Z_1+Z_2^*Z_2=I$, it can be extended to the matrix $\begin{pmatrix} M & U & K_1 \\ Z_1 & V_1 & K_2 \\ Z_2 & V_2 & K_3 \end{pmatrix}$ and apply (C-S decomposition) in [18, p. 77] and [19, p. 37], there exist unitary matrices U_1, U_2, U_3, P, Q, R and diagonal matrices $\Gamma>0$, $\Sigma\geq 0$ and $\Psi\geq 0$ such that:

$$\begin{pmatrix} M & U & K_1 \\ Z_1 & V_1 & K_2 \\ Z_2 & V_2 & K_3 \end{pmatrix}$$
$$=\begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{pmatrix}\begin{pmatrix} \sqrt{2}\Gamma & -\Sigma & -\dfrac{\Psi}{\sqrt{2}\Gamma} \\ \Sigma & \sqrt{2}\Gamma & 0 \\ \Psi & 0 & 1 \end{pmatrix}\begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{pmatrix}$$

where $\left(2\Gamma^2+\Sigma^2+\Psi^2\right)=I$.

So $M=\sqrt{2}U_1\Gamma P$, $Z_1=U_2\Sigma P$ and $Z_2=U_3\Psi P$

Then (2.1) can be rewritten as follows:

$$A=2P^*\Gamma^2PU_2\Sigma P\quad ,\quad B=2P^*\Gamma^2PU_3\Psi P$$

Conversely, suppose that A and B have the decomposition (2.3). Let $X=2P^*\Gamma^2P$. Then

$$\therefore\ X + A^{*}X^{-2}A + B^{*}X^{-2}B = 2P^{*}\Gamma^{2}P$$
$$+\left(2P^{*}\Gamma^{2}PU_{2}\Sigma P\right)^{*}\left(2P^{*}\Gamma^{2}P\right)^{-2}\left(2P^{*}\Gamma^{2}PU_{2}\Sigma P\right)$$
$$+\left(2P^{*}\Gamma^{2}PU_{3}\Psi P\right)^{*}\left(2P^{*}\Gamma^{2}P\right)^{-2}\left(2P^{*}\Gamma^{2}PU_{3}\Psi P\right)$$
$$= 2P^{*}\Gamma^{2}P + P^{*}\Sigma U_{2}^{*}\left(P^{*}\Gamma^{2}P\right)\left(P^{*}\Gamma^{2}P\right)^{-2}\left(P^{*}\Gamma^{2}P\right)U_{2}\Sigma P$$
$$+P^{*}\Psi U_{3}^{*}\left(P^{*}\Gamma^{2}P\right)\left(P^{*}\Gamma^{2}P\right)^{-2}\left(P^{*}\Gamma^{2}P\right)U_{3}\Psi P$$
$$= 2P^{*}\Gamma^{2}P + P^{*}\Sigma^{2}P + P^{*}\Psi^{2}P$$
$$= P^{*}\left(2\Gamma^{2}+\Sigma^{2}+\Psi^{2}\right)P = I$$

this shows that X is a solution to the matrix Eq. (1.1).

3. MAIN RESULTS

In this section, we propose some properties of the equation (1.1) and we present relations between the solution X and the matrices A and B.

Theorem 3.1

If the matrix equation (1.1) has a positive definite solution X, then $\| A+B \| < 1$

Proof:

Suppose that Eq. (1.1) has a solution., then by Theorem 2.2, A and B have the decomposition (2.3).Then

$$\| A+B \| = \left\| 2P^{*}\Gamma^{2}PU_{2}\Sigma P + 2P^{*}\Gamma^{2}PU_{3}\Psi P \right\|$$
$$\le \left\| 2P^{*}\Gamma^{2}PU_{2}\Sigma P \right\| + \left\| 2P^{*}\Gamma^{2}PU_{3}\Psi P \right\|$$
$$\le 2\left\| P^{*}\Gamma^{2}P \right\| \| U_{2} \| \| \Sigma \| \| P \|$$
$$+2\left\| P^{*}\Gamma^{2}P \right\| \| U_{3} \| \| \Psi \| \| P \|$$

Since the matrices U_2 ,U_3 and P are unitary,

$$\therefore \| A+B \| \le 2\left\| P^{*}\Gamma^{2}P \right\| \| \Sigma \| + 2\left\| P^{*}\Gamma^{2}P \right\| \| \Psi \|$$
$$\le 2\left\| \Gamma^{2} \right\| \| \Sigma \| + 2\left\| \Gamma^{2} \right\| \| \Psi \|$$
$$= 2\left\| \Gamma^{2} \right\| \left\{ \| \Sigma \| + \| \Psi \| \right\} < 1$$

The last inequality follows from the fact $\left(2\Gamma^{2}+\Sigma^{2}+\Psi^{2}\right) = I$, which yields that $\| \Gamma \| \le \frac{1}{2}$, $\| \Sigma \| < 1$ and $\| \Psi \| < 1$

Theorem 3.2

Suppose that the matrix equation (1.1) has a positive definite solution X , then the following hold:

$(i)\ X^{2} > AA^{*} + BB^{*}$

$(ii)\ r\left(A+B\right) \le \frac{1}{2}$

$(iii)\ r\left(X^{\frac{-1}{2}}A + X^{\frac{-1}{2}}B - A^{*}X^{\frac{-1}{2}} - B^{*}X^{\frac{-1}{2}} \right) \le 1$

$(iv)\ r\left(X^{\frac{-1}{2}}A + X^{\frac{-1}{2}}B + A^{*}X^{\frac{-1}{2}} + B^{*}X^{\frac{-1}{2}} \right) \le 1$

Proof:

(i): By Theorem 2.1 we have, $X = (M^{*}M)$ and

$A = (M^{*}M)Z_{1}$, $B = (M^{*}M)Z_{2}$

Then

$$X^{2} - AA^{*} - BB^{*} = (M^{*}M)^{2} - (M^{*}M)Z_{1}Z_{1}^{*}(M^{*}M)$$
$$-(M^{*}M)Z_{2}Z_{2}^{*}(M^{*}M)$$
$$= (M^{*}M)\left(I - Z_{1}Z_{1}^{*} - Z_{2}Z_{2}^{*}\right)(M^{*}M) > 0$$

since $\lambda(Z_{i}Z_{i}^{*}) = \lambda(Z_{i}^{*}Z_{i}),\ i = 1,2$ and

$I - \sum_{i=1}^{2} Z_{i}^{*}Z_{i} = M^{*}M > 0$, so $I - Z_{1}Z_{1}^{*} - Z_{2}Z_{2}^{*} > 0$.

Thus, part (i) is proved.

(ii): Using Theorem 3.1 we get,

$$\lambda(A+B) = \lambda\left(2P^{*}\Gamma^{2}PU_{2}\Sigma P + 2P^{*}\Gamma^{2}PU_{3}\Psi P\right)$$

$$r(A+B) = \max_{i}\left| \lambda_{i}\left(2P^{*}\Gamma^{2}PU_{2}\Sigma P + 2P^{*}\Gamma^{2}PU_{3}\Psi P\right)\right|$$
$$\le \left\| 2P^{*}\Gamma^{2}PU_{2}\Sigma P + 2P^{*}\Gamma^{2}PU_{3}\Psi P \right\|$$
$$\le \left\| 2P^{*}\Gamma^{2}PU_{2}\Sigma P \right\| + \left\| 2P^{*}\Gamma^{2}PU_{3}\Psi P \right\|$$
$$= 2\left\| P^{*}\Sigma\Gamma^{2}P \right\| + 2\left\| P^{*}\Psi\Gamma^{2}P \right\|$$
$$= 2\left\| \Sigma\Gamma^{2} \right\| + 2\left\| \Psi\Gamma^{2} \right\|$$
$$\le 2\| \Sigma\Gamma \| \| \Gamma \| + 2\| \Psi\Gamma \| \| \Gamma \|$$
$$\le \| \Sigma\Gamma \| + \| \Psi\Gamma \|$$

Let $\Sigma = diag(\sigma_{i})$, $\Gamma = diag(\gamma_{i})$, $\Psi = diag(\psi_{i})$.

Then $\sigma_{i}^{2} + 2\gamma_{i}^{2} + \psi_{i}^{2} = 1$ and $\| \Gamma \| \le \frac{1}{2}$. Thus

$$\begin{aligned} r(A+B) &\le \|\Sigma\Gamma\| + \|\Psi\Gamma\| \\ &= \max_i\{\sigma_i\gamma_i\} + \max_i\{\psi_i\gamma_i\} \\ &\le \max_i \frac{\sigma_i^2 + 2\gamma_i^2 + \psi_i^2}{2} = \frac{1}{2} \end{aligned}$$

(iii): Appling Lemma 6 in [20], we get

$$\begin{aligned} & r\left(X^{\frac{-1}{2}}A + X^{\frac{-1}{2}}B - A^*X^{\frac{-1}{2}} - B^*X^{\frac{-1}{2}}\right) \\ &= r\left(\begin{array}{l}(M^*M)^{\frac{-1}{2}}\left((M^*M)Z_1\right) + (M^*M)^{\frac{-1}{2}}\left((M^*M)Z_2\right) \\ -\left(Z_1^*(M^*M)\right)(M^*M)^{\frac{-1}{2}} - \left(Z_2^*(M^*M)\right)(M^*M)^{\frac{-1}{2}}\end{array}\right) \\ &= r\left(\begin{array}{l}(M^*M)^{\frac{1}{2}}Z_1 + (M^*M)^{\frac{1}{2}}Z_2 \\ -Z_1^*(M^*M)^{\frac{1}{2}} - Z_2^*(M^*M)^{\frac{1}{2}}\end{array}\right) \\ &\le r\left(M^*M + Z_1^*Z_1 + Z_2^*Z_2\right) \\ &= r(I) = 1 \end{aligned}$$

(iv):

$$\begin{aligned} & I \pm \left(X^{\frac{-1}{2}}A + X^{\frac{-1}{2}}B + A^*X^{\frac{-1}{2}} + B^*X^{\frac{-1}{2}}\right) \\ &= M^*M + Z_1^*Z_1 + Z_2^*Z_2 \pm \left(\begin{array}{l}(M^*M)^{\frac{1}{2}}Z_1 + (M^*M)^{\frac{1}{2}}Z_2 \\ +Z_1^*(M^*M)^{\frac{1}{2}} + Z_2^*(M^*M)^{\frac{1}{2}}\end{array}\right) \\ &= \left(M^*M \pm Z_1^*Z_1 \pm Z_2^*Z_2\right)^*\left(M^*M \pm Z_1^*Z_1 \pm Z_2^*Z_2\right) \ge 0 \end{aligned}$$

Therefore,

$$r\left(X^{\frac{-1}{2}}A + X^{\frac{-1}{2}}B + A^*X^{\frac{-1}{2}} + B^*X^{\frac{-1}{2}}\right) \le 1.$$

4- CONCLUSION

In this paper both necessary and sufficient conditions for the nonlinear matrix equation $X + A^*X^{-2}A + B^*X^{-2}B = I$ to have a positive definite solution X are derived, where A and B are $n \times n$ nonsingular complex matrices. Some properties on the matrices A and B are presented. Also, relations between the positive definite solution X and the matrices A and B are given.

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