ON NONLINEAR INTEGRAL INEQUALITIES IN ONE VARIABLE AND APPLICATIONS

S. G. Latpate
Nowrosjee Wadia College, Pune-411001.

Abstract In this paper, we establish integral inequalities in one variable which are useful to study differential and integral equations.

Keywords: Integral inequalities, Integral equations, Bounded Functions

1. Introduction

Integral inequalities involving functions of one independent variable, which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of nonlinear differential and integral equations. In recent years integral inequalities have received considerable attention because of the important applications to a variety of problems in diverse fields of differential and integral equations.

In the study of the qualitative behavior of solutions of differential and integral equations, integral inequalities established by Gronwall [7], Bellman [2] and Pachpatte's [9, 10] play a fundamental role. The generalizations and variants of these inequalities are further studied by many mathematicians, see [4, 6, 8, 9, 10] and references are therein.

In this paper, we extend and improve some of the results to obtain generalizations and variants of Pachpatte's integral inequalities which find significant applications in the study of various classes of differential and integral equations.

The following Lemma is useful in our main result.

Lemma 1. [11]*: Assume that* $a \geq 0, p \geq 1$, *then*

$$a^{\frac{1}{p}} \leq \frac{1}{p} k^{\frac{1-p}{p}} a + \frac{p-1}{p} k^{\frac{1}{p}}, \tag{0.1}$$

for any $k > 0$.

2. Main Results

In this section, we state and prove integral inequalities to obtain explicit bound on solutions of certain nonlinear differential and integral equations

Theorem 2. *Let* $u(t), f_1(t), f_2(t), g(t)$ *be nonnegative real valued continuous functions on* $\mathbb{R}_+ = [0, \infty)$ *and if*

$$u^p(t) \leq u_0 + \int_0^t [f_1(s)u^p(s) + g(s)]\, ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1)u(s_1) ds_1 \right) ds, \tag{0.2}$$

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then

$$u^p(t) \le \left(u_0 + \int_0^t [g(s) + m_2 f_2(s)]\, ds\right) \exp\left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds\right), \tag{0.3}$$

where $p \ge 1,\ k > 0,\ m_1 = \frac{1}{p} k^{\frac{1-p}{p}}$ *and* $m_2 = \frac{p-1}{p} k^{\frac{1}{p}}$.

Proof. Define a function $z(t)$ by

$$z(t) = u_0 + \int_0^t [f_1(s) u^p(s) + g(s)]\, ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1) u(s_1) ds_1\right) ds,$$

then $u(t) \le z^{\frac{1}{p}}(t),\ z(0) = u_0$ and

$$\begin{aligned} z'(t) &= f_1(t) u^p(t) + g(t) + f_1(t) \left(\int_0^t [f_2(s_1) u(s_1) ds_1\right) \\ &\le g(t) + f_1(t) \left(z(t) + \int_0^t f_2(s_1) z^{\frac{1}{p}}(s_1) ds_1\right). \end{aligned} \tag{0.4}$$

From Lemma 1 and equation (0.4), we have

$$z'(t) \le g(t) + f_1(t) \left(z(t) + \int_0^t f_2(s_1) \left(m_1 z(s_1) + m_2\right) ds_1\right).$$

Let $v(t)$ be

$$v(t) = z(t) + \int_0^t f_2(s_1) \left(m_1 z(s_1) + m_2\right) ds_1,$$

then $u^p(t) \le z(t) \le v(t),\ v(0) = z(0) = u_0$ and

$$\begin{aligned} v'(t) &= z'(t) + f_2(t) \left(m_1 z(t) + m_2\right) \\ &\le z'(t) + f_2(t) \left(m_1 v(t) + m_2\right) \\ &\le g(t) + f_1(t) v(t) + f_2(t) \left(m_1 v(t) + m_2\right) \end{aligned}$$

$$\text{i.e. } \left[\frac{v(t)}{\exp(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds}\right]' \le g(t) + m_2 f_2(t). \tag{0.5}$$

Set $t = s$; in equation (0.5) and integrate with respect to s from 0 to t, to obtain the estimate

$$v(t) \le \left(u_0 + \int_0^t [g(s) + m_2 f_2(s)]\, ds\right) \exp\left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds\right). \tag{0.6}$$

Using the fact that $u^p(t) \le z(t) \le v(t)$ and above inequality (0.6) we obtain the desired bound for $u(t)$ given by (0.3). □

Theorem 3. *Let* $u(t), f_1(t), f_2(t), g(t)$ *be nonnegative real valued continuous functions on* $\mathbb{R}_+$ *and if*

$$u^p(t) \le u_0 + \int_0^t f_1(s) u^p(s) ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1) u(s_1) + g(s_1) ds_1\right) ds$$

then

$$u^p(t) \le \left(u_0 + \int_0^t [g(s) + m_2 f_2(s)]\, ds\right) \exp\left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds\right),$$

where p, m_1, m_2 *are defined as same in Theorem 2.*

Proof. The Proof of Theorem 3 can be completed by following the same arguments as in the proof of Theorem 2 with suitable modifications. □

Corollary 4. *Let $u(t), f_1(t), f_2(t), g(t), n(t)$ be nonnegative real valued continuous functions on $\mathbb{R}_+$, $n(t)$ be nondecreasing and $n(t) \geq 1$, on $\mathbb{R}_+$ and if*

$$u^p(t) \leq n(t) + \int_0^t [f_1(s)u^p(s) + g(s)]\, ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1)u(s_1)ds_1 \right) ds \tag{0.7}$$

then

$$u^p(t) \leq n^p(t) \left(1 + \int_0^t [g(s) + m_2 f_2(s)]\, ds \right) \exp \left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds \right), \tag{0.8}$$

where p, m_1, m_2 are defined as same in Theorem 2.

Proof. Since $n(t)$ is nondecreasing and $n(t) \geq 1$. From equation (0.7), we have

$$\frac{u^p(t)}{n^p(t)} \leq 1 + \int_0^t \left[f_1(s) \frac{u^p(s)}{n^p(s)} + g(s) \right] ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1) \frac{u(s_1)}{n(s_1)} ds_1 \right) ds, \tag{0.9}$$

Applying Theorem (2) to the equation (0.9), we obtain (0.8). This completes the proof. □

Theorem 5. *Let $u(t), f_1(t), f_2(t), g_1(t), g_2(t)$ be nonnegative real valued continuous functions on $\mathbb{R}_+$ and if*

$$u^p(t) \leq u_0 + \int_0^t [f_1(s)u^p(s) + g_1(s)]\, ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1)u(s_1) + g_2(s_1)ds_1 \right) ds, \tag{0.10}$$

then

$$u^p(t) \leq \left(u_0 + \int_0^t [g_1(s) + g_2(s) + m_2 f_2(s)]\, ds \right) \exp \left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds \right), \tag{0.11}$$

where p, m_1, m_2 are defined as same in Theorem 2.

Proof. Let right hand side of (0.10) be $z(t)$, then $u^p(t) \leq z(t), z(0) = u_0$ and

$$z'(t) \leq g_1(t) + f_1(t) \left(z(t) + \int_0^t \left[f_2(s_1) z^{\frac{1}{p}}(s_1) + g_2(s_1) \right] ds_1 \right). \tag{0.12}$$

From Lemma 1 and equation (0.12), we have

$$z'(t) \leq g_1(t) + f_1(t) \left(z(t) + \int_0^t [f_2(s_1)\,(m_1 z(s_1) + m_2) + g_2(s_1)]\, ds_1 \right).$$

Define a function $v(t)$ by

$$v(t) = z(t) + \int_0^t [f_2(s_1)\,(m_1 z(s_1) + m_2) + g_2(s_1)]\, ds_1$$

then $u^p(t) \leq z(t) \leq v(t)$, $v(0) = z(0) = u_0$ and

$$\begin{aligned} v'(t) &= z'(t) + f_2(t)\,(m_1 z(t) + m_2) + g_2(t) \\ &\leq z'(t) + f_2(t)\,(m_1 v(t) + m_2) + g_2(t) \\ &\leq g_1(t) + f_1(t)v(t) + f_2(t)\,(m_1 v(t) + m_2) + g_2(t), \end{aligned}$$

or, equivalently,

$$\left[\frac{v(t)}{\exp(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds} \right]' \leq g_1(t) + g_2(t) + m_2 f_2(t). \tag{0.13}$$

Set $t = s$; in equation (0.13) and integrate with respect to s from 0 to t, we get obtain the estimate

$$v(t) \leq \left(u_0 + \int_0^t [g_1(s) + g_2(s) + m_2 f_2(s)]\, ds \right) \exp \left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds \right). \tag{0.14}$$

Using the fact that $u^p(t) \leq z(t) \leq v(t)$ and above inequality (0.14) we obtain the desired bound for $u(t)$ given by (0.11). □

Corollary 6. *Let $u(t), f_1(t), f_2(t), g(t), n(t)$ be nonnegative real valued continuous functions on $\mathbb{R}_+$, $n(t)$ be nondecreasing and $n(t) \geq 1$ on $\mathbb{R}_+$, and if*

$$u^p(t) \leq n(t) + \int_0^t [f_1(s)u^p(s) + g_1(s)]\, ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1)u(s_1) + g_2(s_1)ds_1 \right) ds, \tag{0.15}$$

then

$$u^p(t) \leq n^p(t) \left(1 + \int_0^t [g_1(s) + g_2(s) + m_2 f_2(s)]\, ds \right) \exp\left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds \right), \tag{0.16}$$

where p, m_1, m_2 are defined as same in Theorem 2.

Proof. Since $n(t)$ is nondecreasing and $n(t) \geq 1$. From equation (0.15), we have

$$\frac{u^p(t)}{n^p(t)} \leq 1 + \int_0^t \left[f_1(s)\frac{u^p(s)}{n^p(s)} + g_1(s) \right] ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1)\frac{u(s_1)}{n(s_1)} + g_2(s_1)ds_1 \right) ds. \tag{0.17}$$

Now applying Theorem 5 to equation (0.17), we get(0.16). □

Corollary 7. *Let $u(t), f_1(t), f_2(t), g(t)$ be nonnegative real valued continuous functions on $\mathbb{R}_+$ and if*

$$u^p(t) \leq u_0 + \int_0^t [f_1(s)u^p(s) + g(s)]\, ds + \int_0^t f_1(s) \left(\int_0^s [f_2(s_1)u(s_1) + g(s_1)ds_1 \right) ds, \tag{0.18}$$

then

$$u^p(t) \leq \left(u_0 + \int_0^t [2g(s) + m_2 f_2(s)]\, ds \right) \exp\left(\int_0^t [f_1(s) + m_1 f_2(s)]\, ds \right), \tag{0.19}$$

where p, m_1, m_2 are defined as same in Theorem 2.

Proof. Proof follows if we take $g_1(t) = g_2(t)$ in Theorem 5 □

3. Applications

In this section, we present applications of some of the inequalities established in previous sections.

Example 1. *We calculate the explicit bound on the solution of the nonlinear integral equation of the form:*

$$u^2(t) = 2 + \int_0^t \left[e^s u^2(s) + \frac{1}{1+s} \right] ds + \int_0^t e^s \left(\int_0^s \frac{1}{1+s_1^2} u(s_1) ds_1 \right) ds \tag{0.20}$$

where $u(t)$ are defined as in Theorem 2 and we assume that every solution $u(t)$ of (0.20) exists on $\mathbb{R}_+$.

Applying Theorem 2 to the equation (0.20), we have

$$\begin{aligned} u^2(t) &\leq \left(2 + \int_0^t \left[\frac{1}{1+s} + \frac{1}{2}k^{\frac{1}{2}} \frac{1}{1+s_1^2} \right] ds \right) \exp\left(\int_0^t \left[e^s + \frac{1}{2}k^{\frac{-1}{2}} \frac{1}{1+s^2} \right] ds \right), \\ &= \left(2 + \log(1+t) + \frac{\sqrt{k}}{2} \tan^{-1} t \right) \exp\left(e^t - 1 + \frac{1}{2\sqrt{k}} \tan^{-1} t \right). \end{aligned} \tag{0.21}$$

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