# An Analysis of the Flow of Blood through Narrow Tapered Tubes

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***Abstract**: In the present analysis, we have studied the flow of blood through a uniform tapered tube assuming it obeys Bingham fluid model. The wall shear stress and pressure gradient have been obtained. The variations of pressure gradient and wall shear stress are shown in tables.*

**Keywords:** *Bingham fluid model, Pressure gradient, Wall shear stress, Tapered, Tapered Tube, Tapered angle.*

**I. Introduction:** The behavior of blood flow is mainly due to the suspension of red cells in an aqueous phase of the complex structure is called plasma. Due to presence of the suspended particle in plasma, there have been number of attempts to explain the anomalous behavior of blood by proposing different theoretical Newtonian and Non- Newtonian fluid models. Iida and Murata [7] studied pulsatile blood flow through small vessels by assuming Herschel –Bulkley fluid model of blood Ariman et al. [1], Bugliarello and Sevilla [ 3]. The idea of a tapered tube model of a blood vessel was given by Womersley [10], Block [2], Jeffords and Knisely [8] have investigated that all the vessels which carry blood towards the tissues should be considered as long , slowely tapering cones rather than cylinders. Charm and Kurland [4] have examined flow properties of blood flowing through non- uniform capillary tubes. Chaturni and Phalhad [5] have studied a steady laminar flow of blood in a uniform tapered tube by assuming blood as polar fluid. Dwivedi et al. [6], Nishmura [9] have studied the steady blood flow in tapered tube by taking different models of blood flow through tapered arteries with stenosis and S. Chakravarthy , P.K Mandal [11] studied two dimensional blood flow through tapered arteries under stenotic condition . The expression for wall shear stress , pressure drop , total angular and axial velocities have been calculated.

**II. Mathematical Formulation:** We consider a steady laminar flow of incompressible viscous Non-Newtonian fluid model in a uniformly tapered tube of circular cross-section.

The radius of the tapered tube $R(z)$ is given by

$$R(z) = R_0 - z\tan\phi \tag{1}$$

Where $R_0$ is the tube radius at $z = 0$, $\phi$ is tapered angle and z is the axis of the tapered tube.

**III. The Governing Equation:** The governing equation in cylindrical co-ordinate system $(r, \theta\ z)$, which mathematically describe the laminar flow problem of an incompressible viscous fluid.

The continuity equation is given by

$$\frac{\partial V_z}{\partial z} + \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0 \tag{2}$$

Momentum equation is given by $\rho\frac{DV_z}{Dt} = -\frac{\partial P}{\partial z} + \frac{\partial}{\partial z}\left(2\eta\frac{\partial V_z}{\partial z}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\eta\left\{\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right\}\right]$ +

$$\frac{\partial}{\partial r}\left[\eta\left\{\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right\}\right] + \frac{\eta}{r}\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right) \tag{3}$$

Where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r\frac{\partial}{\partial r}V_z\frac{\partial}{\partial z} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}$$

If the motion is steady and axisymmetric, then

$$\frac{\partial}{\partial t} = 0\ ,\quad \frac{\partial}{\partial \theta} = 0\ ,\quad V_r = 0 = V_\theta,\quad V_z = V(r) \tag{4}$$

Using equation (4) in equation (3)

$$0 = -\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{5}$$

Where P is the pressure, $\tau_{rz} = \eta \frac{\partial V}{\partial r}$ is the shear stress normal to r in $z$ direction and V is the axial velocity.

**IV. Constitutive Equation:** The constitutive equation for the shear stress $\tau$ and strain rate $\gamma$ is given by

$$\tau = \tau_0 + \eta\dot{\gamma} \;; \qquad \tau \ge \tau_0$$

$$\dot{\gamma} = 0 \;; \qquad \tau \le \tau_0 \qquad (6)$$

Where $\tau_0$ is the yield stress, $\eta$ is the coefficient of viscosity and $\gamma$ is the shear strain rate.

**V. The Boundary Condition:** The boundary conditions are given by

$$V = 0, \quad \text{at} \quad r = R(z) \qquad (7)$$

$$\tau_{rz} = \tau_w, \quad \text{at} \quad r = R(z) \qquad (8)$$

$$V = V_P, \quad \text{at} \quad r = R_P \qquad (9)$$

$$\tau_{rz} \text{ is finite at } \quad r = 0 \qquad (10)$$

Where $R_P$ is the plug radius and $V_P$ is the plug velocity. Integrating equation (5) and using boundary condition

(10), we get

$$\tau_{rz} = \frac{r}{2}\frac{\partial P}{\partial z} \qquad (11)$$

With the help of constitutive equation, we get the velocity equation as

$$\frac{\partial V}{\partial r} = \frac{1}{\eta}\left(\frac{\partial P}{\partial z}\frac{r}{2} - \tau_0\right); \qquad R_P \le r \le R(z) \qquad (12)$$

$$\frac{dV_P}{dr} = 0; \qquad 0 \le r \le R_P \qquad (13)$$

The plug flow exists whenever the shear stress does not exceed yield stress.
Solving equation (12) and (13), we get

$$V = \frac{\tau_\omega(z)}{2\eta}R(z)\left[1 - \frac{r^2}{R^2(z)} - 2\beta\left(1 - \frac{r}{R(z)}\right)\right] \qquad (14)$$

$$V_P = \frac{\tau_\omega(z)}{2\eta}R(z)(1-\beta)^2 \qquad (15)$$

Where $\beta = \frac{\tau_0}{\tau_w(z)}$

The volume flow rate Q is given by

$$Q = Q_1 + Q_2 \qquad (16)$$

Where $Q_1 = \int_0^{R_P} 2\pi V_P r\, dr = \pi V_P R_P{}^2$ (17)

and $Q_2 = \int_{R_P}^{R(z)} 2\pi V r dr$ (18)

Now substitute the values of $V_P$ and V from equations (14) and (15) in equations (17) and (18), we get $Q = \dfrac{\pi\tau_\omega(z)R^3(z)\left(1-\frac{4}{3}\beta\right)}{4\eta}$ (19) Now from equations (11) and (19) with boundary condition (8), the pressure gradient is obtained as $\dfrac{\partial P}{\partial z} = \dfrac{8\eta Q}{\pi R^4(z)}\left(1+\frac{4}{3}\beta\right)$ (20) From equation (19), we have the shear stress at the wall as

$$\tau_w(z) = \frac{4\eta Q}{\pi R^3(z)}\left(1+\frac{4}{3}\beta\right) \quad (21)$$

Now using equation (20) and (21), we have $\tau_\omega(z) = \dfrac{R(z)}{2}\dfrac{\partial P}{\partial z}$ (22) From the equation (20) and (21), we obtained that both the pressure gradient and the shear stress at the wall increases when $R(z)$ decreases.

**VI. Table1 :** Variation of pressure gradient with flow rate for different value of tapered angle $z = 0.01,\ R_0 = 0.01,\ \beta = 1.0$

| *Q* | $dP/dz$ | | | | |
|---|---|---|---|---|---|
| | $\phi = 1.0$ | $\phi = 1.2$ | $\phi = 1.4$ | $\phi = 1.6$ | $\phi = 1.8$ |
| 0.01 | 0.063 | 0.064 | 0.065 | 0.066 | 0.067 |
| 0.02 | 0.127 | 0.129 | 0.131 | 0.133 | 0.135 |
| 0.03 | 0.191 | 0.193 | 0.197 | 0.199 | 0.203 |
| 0.04 | 0.255 | 0.258 | 0.263 | 0.266 | 0.271 |
| 0.05 | 0.318 | 0.323 | 0.328 | 0.333 | 0.339 |
| 0.06 | 0.382 | 0.387 | 0.394 | 0.399 | 0.407 |

**VII. Table 2:** Variation of pressure gradient with axial distance for different value of flow rates $R_0 = 0.01, \phi = 1.4^\circ, \beta = 2.0$

| *Z* | $dP/dz$ | | | | | |
|---|---|---|---|---|---|---|
| | $Q_1 = 0.01$ | $Q_2 = 0.02$ | $Q_3 = 0.03$ | $Q_4 = 0.04$ | $Q_5 = 0.05$ | $Q_6 = 0.06$ |
| 0.00 | 0.093 | 0.186 | 0.280 | 0.373 | 0.467 | 0.560 |
| 0.02 | 0.115 | 0.231 | 0.346 | 0.462 | 0.578 | 0.693 |
| 0.04 | 0.151 | 0.303 | 0.455 | 0.606 | 0.758 | 0.909 |
| 0.06 | 0.220 | 0.441 | 0.661 | 0.882 | 1.102 | 1.323 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.08 | 0.409 | 0.805 | 1.208 | 1.610 | 2.013 | 2.416 |
| 0.10 | 2.340 | 4.680 | 7.021 | 9.360 | 11.70 | 14.04 |

**VIII. Table 3. (** For Newtonian Fluid **)** Variation of wall shear stress with flow rate for different tapered angle $z = 0.1,\ R_0 = 0.01,\ \beta = 0.0$

| $Q$ | $dP/dz$ | | | | |
|---|---|---|---|---|---|
| | $\phi = 1.0$ | $\phi = 1.2$ | $\phi = 1.4$ | $\phi = 1.6$ | $\phi = 1.8$ |
| 0.01 | 0.084 | 0.155 | 0.640 | -0.318 | -0.099 |
| 0.02 | 0.169 | 0.311 | 0.1280 | -0.637 | -0.199 |
| 0.03 | 0.254 | 0.466 | 1.920 | -0.9556 | -0.298 |
| 0.04 | 0.399 | 0.622 | 2.560 | -1.27 | -0.398 |
| 0.05 | 0.424 | 0.778 | 3.200 | -1.593 | -0.497 |
| 0.06 | 0.509 | 0.933 | 3.840 | -1.912 | -0.597 |

**IX. Table 4.** Variation of wall shear stress with flow rate for different tapered angle $z = 0.1,\ R_0 = 0.01,\ \beta = 1.0$

| $Q$ | $\tau_\omega(z)$ | | | | |
|---|---|---|---|---|---|
| | $\phi = 1.0$ | $\phi = 1.2$ | $\phi = 1.4$ | $\phi = 1.6$ | $\phi = 1.8$ |
| 0.01 | 0.121 | 0.148 | 0.212 | 0.313 | 1.346 |
| 0.02 | 0.242 | 0.293 | 0.424 | 0.626 | 2.692 |
| 0.03 | 0.363 | 0.440 | 0.636 | 0.939 | 4.038 |
| 0.04 | 0.485 | 0.587 | 0.849 | 1.252 | 5.384 |
| 0.05 | 0.606 | 0.734 | 1.061 | 1.565 | 6.730 |
| 0.06 | 0.727 | 0.880 | 1.273 | 1.878 | 8.076 |

**X. Table 5.** Variation of wall shear stress with axial distance for different flow rates $R_0 = 0.01,\ \phi = 1.4^\circ\ \beta = 2.0$

| $Z$ | $\tau_\omega(z)$ | | | | |
|---|---|---|---|---|---|
| | $Q_1 = 0.01$ | $Q_2 = 0.02$ | $Q_3 = 0.03$ | $Q_4 = 0.04$ | $Q_5 = 0.05$ |
| 0.00 | 0.0934 | 0.1868 | 0.2802 | 0.3736 | 0.4870 |
| 0.02 | 0.1091 | 0.2182 | 0.3273 | 0.4364 | 0.5455 |
| 0.04 | 0.1312 | 0.2624 | 0.3936 | 0.5248 | 0.6560 |
| 0.06 | 0.1644 | 0.3288 | 0.4932 | 0.6576 | 0.8220 |
| 0.08 | 0.2203 | 0.4406 | 0.6609 | 0.8812 | 1.1015 |

| | | | | | |
|---|---|---|---|---|---|
| 0.10 | 0.3337 | 0.6674 | 1.0011 | 1.3348 | 1.6685 |

**XI. Table 6.** (For Newtonian fluid) Variation of wall shear stress with flow rate for different tapered angle $R_0 = 0.01,\ z = 0.1, \beta = 0.0$

| $Q$ | $\tau_\omega(z)$ | | | | |
|---|---|---|---|---|---|
| | $\phi = 1.0$ | $\phi = 1.2$ | $\phi = 1.4$ | $\phi = 1.6$ | $\phi = 1.8$ |
| 0.01 | 0.0173 | 0.0212 | 0.0303 | 0.0446 | 0.1213 |
| 0.02 | 0.0346 | 0.0424 | 0.0606 | 0.0893 | 0.2426 |
| 0.03 | 0.0519 | 0.0636 | 0.0909 | 0.1340 | 0.3639 |
| 0.04 | 0.0693 | 0.0849 | 0.1213 | 0.1787 | 0.4852 |
| 0.05 | 0.0866 | 0.1061 | 0.1516 | 0.2234 | 0.6066 |

**XII. Result and Discussion:** From the expression of shear stress at wall equation (21) and pressure gradient equation (20), we observed that the wall shear stress and pressure gradient increases with decrease in the radius of the tapered tube.

The variations of pressure gradient and wall shear stress with flow rate for different values of tapered angle are shown in table (1) and (4). We observed that the pressure gradient and wall shear stress increases with increase in flow rate for constant value of tapered angle.

Table (2) and (5) show the variation of pressure gradient and wall shear stress with axial distance for different value of flow rates for constant tapered angle. We observed that pressure gradient increases with increase in axial distance for constant value of flow rates.

For Newtonian fluid $\beta = 0$, the variations of pressure gradient and wall shear stress with flow rate for different value of tapered angle are shown in table (3) and (6). The wall shear stress increases with increase in flow rate but pressure gradient increases with increase in flow rate for constant value of tapered angle 1.0 to 1.4 and after then decreases with increase in flow rate for constant value of tapered angle 1.6 to 1.8.

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