An Introduction to Fractals Geometry

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Abstract: *Fractals were first formally defined by Bonoit Manderbolt in 1980's. A fractal is defined as a rough or fragmented geometric shape that can be subdivided in parts each being a reduced size copy of the whole. Fractals are self-similar across different scales. Mathematically, they are sets obtained through recursion that exhibit interesting dimensional properties. Fractal patterns with various degrees of self-similarity have been studied in images, structures and sounds and found in nature, technology and architecture. They are of particular importance in chaos theory as the graphs of most chaotic processes are fractals. Fractal dimension is used to measure the complexity of objects. The paper overviews the fractals, principles underlying their generation and fractal dimensions.*

Keywords *: self-similar, dimension, recursion*

INTRODUCTION

One of the most intricate images in Mathematics is the Mandelbrot Set, which was discovered by Mandelbrot in 1980. The discovery of this image led to the development of fractal science. Fractal geometry can be considered different to classical geometry in that it does not deal in integer dimensions. More formally, in 1982, Mandelbrot stated that "A fractal is by definition a set for which the Hausdroff-Besicovitch dimension exceeds the topological dimension." Later, he simplified and expanded the definition to: "A fractal is a shape made of parts similar to the whole in some way." Still later, Mandelbrot settled on the use of term fractal dimension as a generic term applicable to all the variants.

Generally, theoretical fractals are infinitely self-similar, iterated and detailed mathematical constructs having fractal dimensions.

A figure is said to be self similar if magnified subsets look–like the whole and to each other. They may not look exactly the same to each other at all scales, but same type of structures must appear on all scales. All self-similar objects may not be fractals but all fractals are self-similar. Fractals have an infinite amount of detail. A smooth curve does not have the property of self-similar, for, on magnifying around any point on the curve, it will eventually look like a line. But this is never the case with a Fractal. There is always more detail as we magnify it. Being a relatively new subject of study, there is no widely accepted definition. One definition is: "A geometric figure or natural object is said to be fractal if

(a) its parts have the same form or structure as the whole, except that they are at a different scale and may be slightly deformed

(b) its form is extremely irregular or interrupted or fragmented, and remains so, whatever the scale of examination;

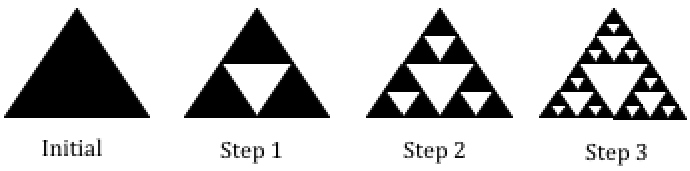
(c) It contains 'distinct elements' whose scales are varied and cover a large range."

(B. Mandelbrot Les objects Fractals1989)

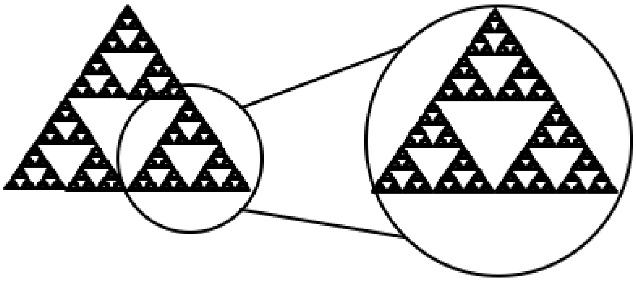
<u>MAKING A FRACTAL</u>

<u>Iterated Fractals: Sierpinski Gasket</u>

The self-similar behaviour can be replicated through recursion: repeating a process over and over.
Example: Connect the mid-points of each side of a filled in equilateral triangle.



The process is repeated over and over, the shape that emerges is called the Sierpinski Gasket. Self-similarity is clearly visible as any piece of gasket looks identical to whole Sierpinski Gasket. In fact, it shows perfect self-similarity. It contains three copies of itself and each copy further contains three copies of itself.



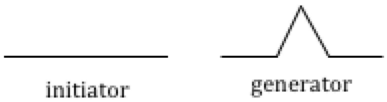
Similar approach can be used to construct other fractals.

Initiators and Generators

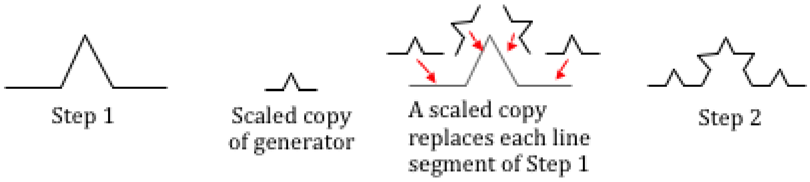
An initiator is a starting shape and a generator is an arranged collection of scaled copies of it. If sequence of shapes is taken (a_n), with a_0 defined as the 'initiator' and a_n defined as the generator applied to a_{n-1} (in above example cutting out the middle triangle by any instances of the initiator within the shape). The final fractal is the limit of this sequence. The initiator-generator approach is a well-defined way of generating self-similar fractal.

The Koch Curve

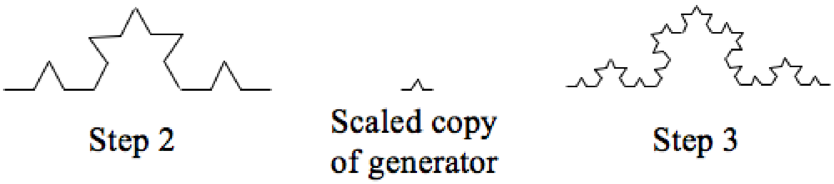
The Koch snow flake or Koch triangle first appeared in a paper published by Swedish Mathematician Neils Fabian Helge Von Koch in 1906. We construct it using initiator-generator concept to construct the iterated fractal where in at each step, every copy of initiator will be replaced with a scaled copy of generator.



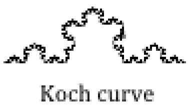
Step 0 – consider a line segment (initiator)
Step 1 – Replace it with the spiked shape generator
Step 2 – Each of the four line segments of step 1 is replaced with a scaled copy of the generator.



This process is repeated to form step 3. Again, each line segment is replaced with a scaled copy of the generator.



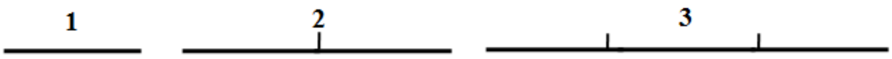
The shape resulting from iterating the process is called Koch Curve.



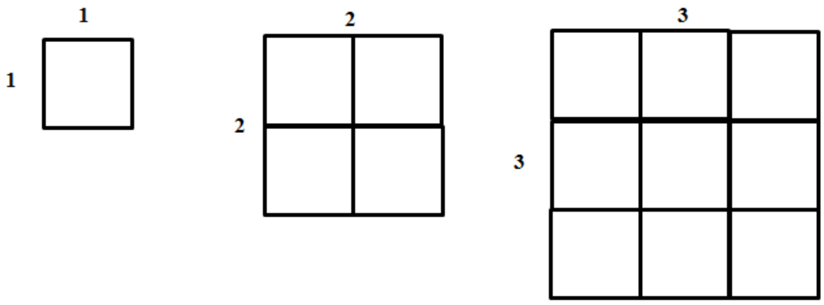
Fractal Dimension

Fractal dimension differs from the classical definition of dimension as it is not based on the cardinality of the basis. However, the two values are equal when the fractal dimension is an integer. It is also known as Housdroff Dimension.

For a one-dimensional shape like line, to scale its length by 2, its 2 copies are needed. Similarly, to scale its length by 3, 3 copies are needed.



In a 2 dimensional figure like square, with side 1 unit, to scale its length and width by 2, four copies of original square are needed. To scale the length and width by 3, nine copies of original square needed.



It can be observed that, number of units or copies needed = scale, for 1-D case
= scale^2 for 2-D case

Scaling dimension relation

For any D-dimensional fractal object, the number of units N of the original shape, needed to scale it by a scaling factor S is given as

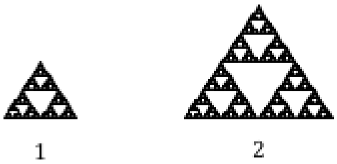
$N = S^D$

Using logarithm $\log N = \log(S^D)$

$$\log N = D\log S$$

$$D = \frac{\log N}{\log S} = \log_S N$$

Example: Dimension of seirpinski Gasket using above relation



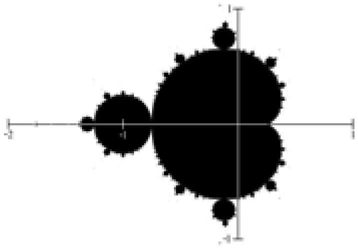
Let the original gasket be defined to have side 1.
The larger gasket has been scaled by a factor of 2 as it is twice as wide and twice as tall.
Also, number of units of original gasket needed = 3
So, $3 = 2^D$
Now, $2^1 = 2$ and $2^2 = 4$ So $1 \le D \le 2$ i.e. D lies between 1 and 2
The gasket is more than 1-Dimensional shape but so much area has been taken away that it is less than 2-Dimentional shape.

$$D = \frac{\log 3}{\log 2} \approx 1.585$$

The dimension of gasket is about 1.585

<u>Mandelbrot Set</u>

The Mandelbrot set is a set of numbers defined using recursive sequence. Consider the function defined by $Z_0 = 0$, $Z_{n+1} = Z_n^2 + 2$
Let $M = \{z \in C | Z_n$ does not diverge to infinity$\}$
M is called the Mandelbrot set.
Conventionally, if Z = 0 and sequence stays close to origin (within 2 units), then number C is part of Mandelbrot set otherwise not.
As a subset of the complex plane, M can be seen.



The above shape is contained in the disc of radius 2, origin lying in the centre of the shape. On zooming the border of the region, border gets quite complicated as one goes further down, various copies (not exact) of the original shape bud off of the main shape. The boundary of the shape exhibits quassi-self-similarity in that positions are not replicating but look very similar to the whole. Fractals of this kind become more interesting when colour is added. It highlights those parts of the set that otherwise are difficult to observe. To create meaningful colouring, number of iterations are counted that are required for a point to get farther than 2 units from origin. For, c = 1 + i, the sequence gets at a distance 2 from origin only after two recursions. While some numbers take tens or hundreds of iterations for sequence to get far from origin. Numbers that get big fast are coloured one shade while that grow slowly are coloured another shade.

It can be continued forever, as one goes deeper; more interesting things are found. The Mandelbrot set having such a simple mathematical definition exhibits immense complexity.

The Julia Set

Julia set is an extension of the Mandelbrot set. The set uses the recursive sequence.

$Z_{n+1} = Z_n^2 + C, \quad Z_0 = d$ Where, C is constant for any particular Julia set and d is the number being tested. A value d is part of the Julia set for C if the sequence does not grow large.

For example: The Julia set for -2 is

$$Z_{n+1} = Z_n^2 - 2, \ Z_0 = d$$

Values of d are picked and tested to determine if it is part of Julia set for -2. If so, colour black the point in the complex plane corresponding with the number d, if not, we can colour the point d based on how slow or fast it grows i.e. based on number of iterations of the recursive sequence for a point to get farther than C units away from Z.

CONCLUSION

The concept of fractal was explored. Infinitely complex shapes can be generated from simple patterns and formulae. A simple well-defined way of doing this is through self-similarity, then moving onto more complex type of fractals. Mandelbrot set and Julia sets using recursive sequences create beautiful colour patterns which become more and more complex as one goes down deeper. Finally, fractal research is a new field of interest. Fractal images are finding their way into computer graphics. Thanks to computers, we can now generate and decode fractals with graphical representations.

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