Solving Fractional Derivative of Some Functions

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Abstract— *In this paper we aim to compare both Riemann-Liouville and Caputo fractional derivatives of some functions like algebraic, trigonometric, logarithmic, exponential and constant.*

Keywords — *Riemann-Liouville Method, Caputo Method, Fractional Derivative, Fractional Integral.*

I. INTRODUCTION

Fractional Calculus is the theory of derivatives and integrals of arbitrary order which unities and generalizes the integer-order differentiation and n-fold integration. The beginning of the fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1695, where the notation for differentiation of non-integer order $\frac{1}{2}$ is discussed. Fractional Calculus is one of the most intensively developing areas of the mathematical analysis as a result of its ircreasing range of applications. Liouville (1832) and Riemann (1876) developed logical definition of fractional operator[3] [4]. The definite integral called the Riemann-Liouville integral for integration of arbitrary order q is defined as

$$_aD_x^{-q}f(x) = \frac{1}{\Gamma(q)}\int_a^x (x-t)^{q-1}f(t)dt \qquad (1)$$

where $Re(q) > 0$ to ensure convergence of integral. When $a = 0$ we have Riemann definition and $a = -\infty$, we have Liouville definition. The function f is such that on $[a, x]$ the integral converges. The notation aD_x^{-q} which denotes the operator of integration of arbitrary order and the corresponding operator aD_x^{q} which denotes the differentiation of arbitrary order was introduced by Davis (1931) [1].

II. RIEMANN-LIOUVILLE METHOD

A. *Preliminaries*

In this section the basic definitions used in fractional calculus are introduced. We start with defining a Riemann-Liouville Method.

Definition II.1 (Fractional Integral) Let $q > 0$ denote a real number. The fractional integral of f of order $-q$ is defined as

$$_aD_x^{-q}f(x) = \frac{1}{\Gamma(q)}\int_0^x (x-t)^{q-1}f(t)dt \qquad (2)$$

The fractional derivative is now defined by applying differentiation a whole number of times to a fractional integral.

Definition II.2 (Fractional Derivative) Let $q > 0$ denote a real number and n be the smallest integer exceeding q. The fractional derivative of f of order q is defined as [3]

$$\frac{d^q f(x)}{dx^q} = \frac{d^n}{dx^n}\left(\frac{d^{-(n-q)}f(x)}{dx^{-(n-q)}}\right) \qquad (3)$$

B. *Caputo Method*

Definition II.3 (Fractional Derivative) Let $q > 0$ denote a real number and n be the smallest integer exceeding q. The fractional derivative of f of order q is defined as[3]

$$_0^cD_x^{-q}f(t) = \int_0^x \frac{f^n(y)}{(x-y)^{q+1-n}}\,dy \qquad (4)$$

III.RESULTS

Using these definitions we compare both the methods for some functions.

A. *Half derivative of x^2 by Riemann Liouville method*

By the definition of fractional integral (2), we find the half integral of $f(x) = x^2$ as follows

$$\frac{d^{-1/2}}{dx^{-1/2}}x^2 = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y^2}{(x-y)^{-1/2+1}}dy$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y^2}{(x-y)^{1/2}}dy$$

put $y = x - \theta \Rightarrow dy = -d\theta$

Evaluating the integrals we get

$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_x^0 \frac{(x-\theta)^2}{\theta^{1/2}}(-d\theta)$$

$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_x^0 \theta^{-1/2}(x-\theta)^2(-d\theta)$$

$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \theta^{-1/2}(x-\theta)^2 d\theta$$

$$=\frac{1}{\sqrt{\pi}}\int_0^x [x^2\theta^{-1/2}-2x\theta^{1/2}+\theta^{3/2}]d\theta$$

$$=\frac{1}{\sqrt{\pi}}\left[2x^{5/2}-\frac{4}{3}x^{5/2}+\frac{2}{5}x^{5/2}\right]$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{30x^{5/2}-20x^{5/2}+6x^{5/2}}{15}\right]$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{16}{15}x^{5/2}\right]$$

B. ***Half derivative of x^2 by Caputo Method***

Using the definition (4) we get,

$$\frac{d^{1/2}x^2}{dx^{1/2}}=\frac{1}{\Gamma(2-1/2)}\int_0^x (-t)^{2-1/2-1}(2)dt$$

$$=\frac{2}{\Gamma(3/2)}\left[\frac{x^{3/2}}{3/2}\right]=\frac{8}{3\sqrt{\pi}}x^{3/2}$$

C. ***Half derivative of $\sin x$ by Riemann-Liouville method***

By the definition of fractional integral, we find the half integral of $f(x)=\sin x$ as follows

$$\frac{d^{-1/2}}{dx^{-1/2}}\sin=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{\sin y}{(x-y)^{-1/2+1}}dy$$

$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{\sin y}{(x-y)^{1/2}}dy$$

Putting $x-y=\theta^2 \Rightarrow dy=-(2\theta)d\theta$

$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\sqrt{x}} \frac{\sin(x-\theta^2)}{\sqrt{\theta^2}}(2\theta)d\theta$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} \sin(x-\theta^2)d\theta$$

$$=\frac{2}{\sqrt{\pi}}\left[\int_0^{\sqrt{x}}(\sin x\cos\theta^2-\cos x\sin\theta^2)\right]d\theta$$

$$=\frac{2}{\sqrt{\pi}}\left[\sin x\int_0^{\sqrt{x}}\cos\theta^2 d\theta-\cos x\int_0^{\sqrt{x}}\sin\theta^2 d\theta\right]$$

Evaluating the integrals become

$$=\frac{2}{\sqrt{\pi}}\left[\sin x\left(\frac{1}{2}\times\sqrt{\frac{\pi}{2}}\right)-\cos x\left(\frac{1}{2}\times\sqrt{\frac{\pi}{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}(\sin x-\cos x)$$

$$\frac{d^{1/2}}{dx^{1/2}}\sin x=\frac{d}{dx}\left(\frac{d^{-1/2}}{dx^{-1/2}}\sin x\right)$$

$$=\frac{1}{\sqrt{2}}\frac{d}{dx}(\sin x-\cos x)$$

$$=\frac{1}{\sqrt{2}}(\cos x+\sin x)$$

D. ***Half derivative of $\sin x$ by Caputo derivative***

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\sin x=\frac{1}{\Gamma(1-1/2)}\int_0^x (x-t)^{1-1/2-1}\frac{d}{dx}(\sin t)dt$$

$$=\frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-1/2}\cos t\,dt$$

Putting $x-t=\theta^2 \Rightarrow dt=-2\theta d\theta$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{x}} \frac{\cos(x - \theta^2)}{\sqrt{\theta^2}} 2\theta d\theta$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^{\sqrt{x}} \cos x \cos \theta^2 + \int_0^{\sqrt{x}} \sin x \sin\theta^2 \right] d\theta$$

$$= \frac{2}{\sqrt{\pi}} \left[\cos x \int_0^{\sqrt{x}} \cos \theta^2 d\theta \right.$$

$$\left. + \sin x \int_0^{\sqrt{x}} \sin \theta^2 d\theta \right]$$

Evaluating the integral we get

$$= \frac{2}{\sqrt{\pi}} \left[\cos x \left(\frac{1}{2} \times \sqrt{\frac{\pi}{2}} \right) + \sin x \left(\frac{1}{2} \times \sqrt{\frac{\pi}{2}} \right) \right]$$

$$= \frac{1}{\sqrt{2}} (\cos x + \sin x)$$

E. ***Fractional integral of $\log x$***

Let $f(x) = \log x$.

Then,

$$D^{-v} \log x = \frac{1}{\Gamma(v)} \int_0^x (x - t)^{v-1} \log t dt \,; v > 0$$

Let $t = xy \Rightarrow dt = xdy$

$$D^{-v} \log x = \frac{1}{\Gamma(v)} \int_0^1 (1$$

$$- xy)^{v-1} \log(xy)\, xdy$$

$$= \frac{x^v}{\Gamma(v)} \int_0^1 (1 - y)^{v-1} \log(xy)\, dy$$

$$= \frac{x^v \log x}{\Gamma(v)} \int_0^1 (1 - y)^{v-1} dy$$

$$+ \frac{x^v}{\Gamma(v)} \int_0^1 (1 - y)^{v-1} \log y dy$$

$$= \frac{x^v \log x}{\Gamma(v+1)} + \frac{x^v}{\Gamma(v)}$$

$$\int_0^1 (1 - y)^{v-1} \log y dy \qquad (5)$$

But from

$$\int_0^1 y^{\mu-1} (1 - y)^{v-1} \log\ y\, dy =$$
$$B(\mu, v)[\psi(\mu) - \psi(\mu + v)], Re\, \mu, v > 0 \quad (6)$$

Let $\mu = 1$ in (6)

$$\int_0^1 (1 - y)^{v-1} \log y dy$$
$$= B(1, v)[\psi(1) - \psi(1 + v)]$$

where B is a beta function. By recurrence relation $\psi(1) = -\gamma$.

$$\int_0^1 (1 - y)^{v-1} \log y dy = \frac{-\gamma - \psi(1 + v)}{v}$$

We can rewrite the equation (5) as:

$$D^{-v} \log x = \frac{x^v \log x}{\Gamma(v+1)} +$$
$$\frac{x^v}{v\Gamma(v)} [-\gamma - \psi(1 + v)]$$

$$D^{-v} \log x = \frac{x^v}{\Gamma(v+1)} [\log x - \gamma$$
$$- \psi(1$$
$$+ v)] \qquad (7)$$

F. Half derivative of log x by Riemann-Liouville method

By using the result (7)

$$dx^{-\frac{1}{2}}d^{-\frac{1}{2}}\log = \frac{x^{\frac{1}{2}}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left[\log x - \gamma - \psi\left(1+\frac{1}{2}\right)\right]$$

$$= \frac{2x^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\left[\log x - \gamma - \psi\left(\frac{3}{2}\right)\right]$$

$$= \frac{2x^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}[\log x - \gamma - (-2\log 2 - \gamma + 2)]$$

$$= \frac{2x^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}[\log x + 2\log 2 - 2]$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\log x = \frac{d}{dx}\left(\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\log x\right)$$

$$= \frac{d}{dx}\left[\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}(\log x + 2\log 2 - 2)\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{1}{2}x^{-\frac{1}{2}}\log x + x^{-\frac{1}{2}} + x^{-\frac{1}{2}}\log 2 - x^{-\frac{1}{2}}\right]$$

$$= \frac{2x^{-\frac{1}{2}}}{\sqrt{\pi}}\left[\frac{1}{2}\log x + \log 2\right]$$

G. Half derivative of $\log x$ by Caputo derivative

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\log x = \frac{1}{\Gamma(1-\frac{1}{2})}\int_0^x (x-t)^{1-\frac{1}{2}-1}f'(\log t)dt$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-t)^{-\frac{1}{2}}\left(\frac{1}{t}\right)dt$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{dt}{t\sqrt{x-t}}$$

$$= \int_0^{\sqrt{x}} \frac{2ydy}{\sqrt{y^2(x-y^2)}}$$

put $x - t = y^2$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} \frac{dy}{(x-y^2)}$$

$$= \frac{2}{\sqrt{\pi x}}\left[\log 2 + \frac{1}{2}\log x\right]$$

H. Half derivative of e^x by Riemann-Liouville method

$$\frac{d^{-1/2}}{dx^{-1/2}}e^x = \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{1/2-1}e^t dt$$

$$= \frac{1}{\Gamma(1/2)}\int_0^x \frac{e^t}{(x-t)^{1/2}}dt$$

putting $y^2 = (x-t) \Rightarrow 2ydy = -dt$

$$= \frac{1}{\Gamma(1/2)}\int_{\sqrt{x}}^0 \frac{e^{x-y^2}}{y}(2ydy)$$

$$= \frac{2}{\Gamma(1/2)}\int_0^{\sqrt{x}} e^{x-y^2}dy$$

$$= \frac{2e^x}{\Gamma(1/2)}\int_0^{\sqrt{x}} e^{-y^2}dy$$

Evaluating the integral we get,

$$\frac{d^{1/2}}{dx^{1/2}}e^x = \frac{2e^x}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = e^x$$

$$\frac{d^{1/2}}{dx^{1/2}}e^x = \frac{d}{dx}\left(\frac{d^{-1/2}}{dx^{-1/2}}e^x\right)$$

$$= \frac{d}{dx}(e^x) = e^x$$

IV. Half derivative of e^x by Caputo derivative

$$\frac{d^{1/2}}{dx^{1/2}}e^x = \frac{1}{\Gamma(1-1/2)}\int_0^x (x - t)^{1-1/2-1} e^t dt$$

$$= \frac{1}{\Gamma(1/2)}\int_0^x \frac{e^t}{(x-t)^{1/2}}dt$$

putting $y^2 = x - t \Rightarrow 2ydy = -dt$

$$= \frac{1}{\Gamma(1/2)}\int_0^{\sqrt{x}} e^{x-y^2}(2dy)$$

$$= \frac{2e^x}{\Gamma(1/2)}\int_0^{\sqrt{x}} e^{-y^2}dy = e^x$$

J. Half derivative of a constant by Riemann-Liouville method

$$\frac{d^{-1/2}}{dx^{-1/2}}1 = \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{1/2-1}1dt$$

$$= \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-1/2}dt$$

Evaluating the integral we get,

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}(2x^{\frac{1}{2}}) = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$\frac{d^{1/2}}{dx^{1/2}}1 = \frac{d}{dx}\left(\frac{d^{-1/2}}{dx^{-1/2}}1\right)$$

$$= \frac{d}{dx}\left(\frac{2\sqrt{x}}{\sqrt{\pi}}\right)$$

$$= \frac{1}{\sqrt{\pi x}}$$

K. Half derivative of a constant by Caputo derivative

$$\frac{d^{1/2}}{dx^{1/2}}1 = \frac{1}{\Gamma(1-1/2)}\int_0^x (x - t)^{1-1/2-1}f'(1)dt$$

$$= \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-1/2}0dt = 0$$

V. Conclusion

In this paper we compare the Riemann and Caputo fractional derivatives of algebraic, trigonometric, logarithmic and exponential functions and find they produce same results. The Riemann and Caputo fractional derivatives differ only in constant function.

VI. References

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