Finite double integrals involving multivariable Aleph-function

Frédéric Ayant

*Teacher in High School , France

Vinod Gill

Department of Mathematics Amity University, Rajasthan, Jaipur-303002, India

ABSTRACT

In this paper we establish two finite double integrals involving the multivariable Aleph-function with general arguments. Our integrals are quite general in character and a number of new integrals can be deduced as particular cases. Several interesting special cases of our main findings have also been mentioned briefly. We will study the particular cases of multivariable I-function, Aleph-function of two variables and I-function of two variables.

KEYWORDS : Aleph-function of several variables, finite double integral, I-function of several variables , Aleph-function of two variables, I-function of two variables.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction and preliminaries.

The multivariable Aleph-function is an extension of the multivariable I-function recently study by C.K. Sharma and Ahmad [5]. The generalized multivariable I-function is a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,n:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} \mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r \end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$

$$\left.\begin{array}{c}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\phi_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\phi_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

where $j = 1$ to r and $k = 1$ to r

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1$ to r , $i = 1$ to R , $i^{(k)} = 1$ to $R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0 \quad \text{, with } k = 1 \text{ to } r \text{ , } i = 1 \text{ to } R \text{ , } i^{(k)} = 1 \text{ to } R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \; ; \; V = m_1, n_1; \cdots ; m_r, n_r \tag{1.6}$$

$$\text{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \quad (1.10)$$

$$D=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \quad (1.11)$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r)=\aleph_{U:W}^{0,n:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{C}\\ \cdot & \\ \cdot & \cdots\\ \cdot & \\ \mathrm{z}_r & \mathrm{B}:\mathrm{D}\end{array}\right) \quad (1.12)$$

2. Required integral

The following known integrals ([2], page 450, Eq.(4)), ([3], page 71, Eq. (3.1.8)) and ([1], page 192, Eq. (46)) will be required during the evaluation of our main results :

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta} sin^{\alpha-1}\theta cos^{\beta-1}\theta \mathrm{d}\theta= e^{(i\pi\alpha)/2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, (Re(\alpha)>0, Re(\beta)>0) \quad (2.1)$$

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta} sin^{\alpha-1}\theta cos^{\beta-1}\theta {}_2F_1[a,b;\beta;e^{i\theta}cos\theta]\mathrm{d}\theta= e^{(i\pi\alpha)/2}\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a)\Gamma(\alpha+\beta-b)}$$

$$(Re(\alpha)>0, Re(\beta)>0, Re(\alpha+\beta-a-b)>0) \quad (2.2)$$

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}P_\upsilon^{(\alpha,\beta)}(1-zx)\mathrm{d}x= \frac{(\alpha+1)_\upsilon\Gamma(\mu)}{\upsilon!}\sum_{k=0}^{\upsilon}\frac{(-\upsilon)_k(1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k\Gamma(\lambda+\mu+k)}\left(\frac{z}{2}\right)^k$$

$$(Re(\lambda)>0, Re(\mu)>0) \quad (2.3)$$

3. Main integrals

First integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}z^{\rho+2\sigma}(1-x)^{\mu-1}(1-y^2)^{\sigma-1}P_\upsilon^{(\alpha,\beta)}(1-zx)$$

$$\aleph\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1}(1-x)^{\sigma_1}y^{\mu_1}(1-y^2)^{\delta_1}z^{\mu_1+2\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r x^{\rho_r}(1-x)^{\sigma_r}y^{\mu_r}(1-y^2)^{\delta_r}z^{\mu_r+2\delta_r}\end{array}\right)\mathrm{d}x\mathrm{d}y= \frac{(\alpha+1)_\upsilon}{\upsilon!}e^{i\pi\rho/2}\sum_{k=0}^{\upsilon}\frac{(-\upsilon)_k(1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k}\left(\frac{z}{2}\right)^k$$

$$\aleph_{U_{42}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c|c} \mathrm{z}_1 e^{i\pi\mu_1/2} & (1\text{-}\mu;\sigma_1,\cdots,\sigma_r),(1-2\sigma;2\delta_1,\cdots,2\delta_r),\\ \cdot & \cdots\\ \cdot & \cdots\\ \cdot & \\ \mathrm{z}_r e^{i\pi\mu_r/2} & (1\text{-}\lambda-\mu-k;\sigma_1+\rho_1,\cdots,\sigma_r+\rho_r),\end{array}\right.$$

$$\left.\begin{matrix} (1\text{-}\lambda - k;\rho_1,\cdots,\rho_r),(1-\rho;\mu_1,\cdots,\mu_r),A:C \\ \cdots \\ \cdots \\ (1\text{-}\rho - 2\sigma;\mu_1+2\delta_1,\cdots,\mu_r+2\delta_r),B:D \end{matrix}\right) \tag{3.1}$$

where $z = \sqrt{1-y^2} + iy$ and $U_{42} = p_i + 4, q_i + 2, \tau_i; R$ (3.2)

The integral (3.1) is valid if the following sets of sufficient conditions are satisfied :

a) $\min\limits_{1 \leqslant i \leqslant r} \{\mu_i, \delta_i, \rho_i, \sigma_i\} > 0, Re(\alpha) > -1, Re(\beta) > -1$

b) $Re(\rho) + \sum_{i=1}^{r} \mu_i \min\limits_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$; $Re(\lambda) + \sum_{i=1}^{r} \rho_i \min\limits_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$

$Re(\mu) + \sum_{i=1}^{r} \sigma_i \min\limits_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$ and $Re(\sigma) + \sum_{i=1}^{r} \delta_i \min\limits_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$

c) $|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.5)

Second integral

$$\int_0^1 \int_0^1 x^{\lambda-1} y^{\gamma-1} z^{\gamma+\delta} (1-x)^{\mu-1} (1-y^2)^{\delta/2-1} P_{\upsilon}^{(\alpha,\beta)}(1-zx) {}_2F_1[a,b;\delta;z\sqrt{1-y^2}]$$

$$\aleph\begin{pmatrix} z_1 x^{\rho_1}(1-x)^{\sigma_1} y^{\mu_1} z^{\mu_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{\rho_r}(1-x)^{\sigma_r} y^{\mu_r} z^{\mu_r} \end{pmatrix} \mathrm{d}x\mathrm{d}y = \frac{(\alpha+1)_{\upsilon}\Gamma(\delta)}{\upsilon!} e^{i\pi\gamma/2} \sum_{k=0}^{\upsilon} \frac{(-\upsilon)_k (1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k} \left(\frac{z}{2}\right)^k$$

$$\aleph_{U_{33}:W}^{0,\mathfrak{n}+3:V}\left(\begin{matrix} z_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_r e^{i\pi\mu_r/2} \end{matrix}\ \middle|\ \begin{matrix} (1\text{-}\gamma;\mu_1,\cdots,\mu_r),(1-\gamma-\delta;\mu_1,\cdots,\mu_r), \\ \cdots \\ \cdots \\ (1\text{-}\gamma-\delta+a;\mu_1,\cdots,\mu_r),(1-\gamma-\delta+b;\mu_1,\cdots,\mu_r), \end{matrix}\right.$$

$$\left.\begin{matrix} (1\text{-}\lambda - k;\rho_1,\cdots,\rho_r),A:C \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-k;\rho_1+\sigma_1,\cdots,\rho_r+\sigma_r),,B:D \end{matrix}\right) \tag{3.3}$$

where $z = \sqrt{1-y^2} + iy$ and $U_{33} = p_i + 3, q_i + 3, \tau_i; R$ (3.4)

The integral (3.3) is valid if the following sets of sufficient conditions are satisfied :

a) $\min_{1\leqslant i\leqslant r}\{\mu_i,\rho_i,\sigma_i\}>0, Re(\alpha)>-1, Re(\beta)>-1, Re(\delta)>0, Re(\gamma+\delta-a-b)>0$

b) $Re(+\sum_{i=1}^{r}\mu_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0\,; Re(\lambda)+\sum_{i=1}^{r}\rho_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$

and $Re(\mu)+\sum_{i=1}^{r}\sigma_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$

c) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5)

Proof

To establish the integral (3.1), we use the Mellin-Barnes type contour integral with the help of (1.1) for the multivariable Aleph-function occuring on the left hand side of (3.1) and changing the order of integration and summation (which is justified under the conditions given with (3.1)) and using (2.1) and (2.3) to evaluate the resulting y-integral and x-integral, respectively. Finally interpreting the resulting contour integral as the multivariable Aleph-function, we get the desired formula (3.1).

The proof of the integral formula (3.3) is similar to that of the first integral with the only difference that here we use the integral (2.2) instead of (2.1).

4. Special cases

Taking $z=2,\mu=\beta+1$ and $\sigma_i\rightarrow 0(i=1,\cdots,r)$ in (3.1) and (3.3). Using Saalschutz's theorem ([3], page 111, Eq. (4.1.12)) to simplify the right-hand side, we get

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}z^{\rho+2\sigma}(1-x)^{\beta}(1-y^2)^{\sigma-1}P_{\upsilon}^{(\alpha,\beta)}(1-2x)$$

$$\aleph\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1}y^{\mu_1}(1-y^2)^{\delta_1}z^{\mu_1+2\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r x^{\rho_r}y^{\mu_r}(1-y^2)^{\delta_r}z^{\mu_r+2\delta_r}\end{array}\right)\mathrm{d}x\mathrm{d}y=\frac{(-)^{\upsilon}\Gamma(\beta+\upsilon+1)}{\upsilon!}e^{i\pi\rho/2}$$

$$\aleph_{U_{43}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c} \mathrm{z}_1 e^{i\pi\mu_1/2}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r e^{i\pi\mu_r/2}\end{array}\left|\begin{array}{c}(1\text{-}\rho;\mu_1,\cdots,\mu_r),(1-2\sigma;2\delta_1,\cdots,2\delta_r),\\ \cdots\\ \cdots\\ (1\text{-}\lambda+\alpha+\upsilon;\rho_1,\cdots,\rho_r),(-\lambda-\beta-\upsilon;\rho_1,\cdots,\rho_r),\end{array}\right.\right.$$

$$\left.\begin{array}{c}(1\text{-}\lambda;\rho_1,\cdots,\rho_r),(1-\lambda+\alpha;\rho_1,\cdots,\rho_r),A:C\\ \cdots\\ \cdots\\ (1\text{-}\rho-2\sigma;\mu_1+2\delta_1,\cdots,\mu_r+2\delta_r),B:D\end{array}\right) \tag{4.1}$$

where $U_{43} = p_i + 4, q_i + 3, \tau_i; R$ and

$$\int_0^1 \int_0^1 x^{\lambda-1} y^{\gamma-1} z^{\gamma+\delta} (1-x)^{\beta} (1-y^2)^{\delta/2-1} P_v^{(\alpha,\beta)}(1-2x) {}_2F_1[a,b;\delta; z\sqrt{1-y^2}] \aleph \begin{pmatrix} z_1 x^{\rho_1} y^{\mu_1} z^{\mu_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{\rho_r} y^{\mu_r} z^{\mu_r} \end{pmatrix} dxdy$$

$$= \frac{(\alpha+1)_v \Gamma(\delta)\Gamma(\beta+v+1)}{v!} e^{i\pi\gamma/2} \aleph_{U_{44}:W}^{0,\mathfrak{n}+4:V} \left(\begin{matrix} z_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_r e^{i\pi\mu_r/2} \end{matrix} \left| \begin{matrix} (1\text{-}\gamma; \mu_1, \cdots, \mu_r), (1-\lambda+\alpha; \rho_1, \cdots, \rho_r), \\ \cdots \\ \cdots \\ (1\text{-}\gamma-\delta+a; \mu_1, \cdots, \mu_r), (-\lambda-\beta-v; \rho_1, \cdots, \rho_r), \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (1\text{-}\gamma-\delta+a+b; \mu_1, \cdots, \mu_r), (-\lambda-\beta; \rho_1, \cdots, \rho_r) A : C \\ \cdots \\ \cdots \\ (1\text{-}\gamma-\delta+b; \mu_1, \cdots, \mu_r), (1-\lambda+\alpha+v; \rho_1, \cdots, \rho_r), B : D \end{matrix} \right) \quad (4.2)$$

where $U_{44} = p_i + 4, q_i + 4, \tau_i; R$

The conditions of validity of (4.1) and (4.2) are the same as these mentioned with (3.1) and (3.3) respectively.

5. Multivariable I-function

If $\tau_i, \tau_{i^{(1)}}, \cdots \tau_{i^{(r)}} \rightarrow 1$, the Aleph-function of several variables reduces to the I-function of several variables. The finite double integrals have been derived in this section for multivariable I-functions defined by Sharma et al [5]. In these section, we have :

First integral

$$\int_0^1 \int_0^1 x^{\lambda-1} y^{\rho-1} z^{\rho+2\sigma} (1-x)^{\mu-1} (1-y^2)^{\sigma-1} P_v^{(\alpha,\beta)}(1-zx)$$

$$I \begin{pmatrix} z_1 x^{\rho_1} (1-x)^{\sigma_1} y^{\mu_1} (1-y^2)^{\delta_1} z^{\mu_1+2\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{\rho_r} (1-x)^{\sigma_r} y^{\mu_r} (1-y^2)^{\delta_r} z^{\mu_r+2\delta_r} \end{pmatrix} dxdy = \frac{(\alpha+1)_v}{v!} e^{i\pi\rho/2} \sum_{k=0}^{v} \frac{(-v)_k (1+\alpha+\beta+v)_k}{k!(\alpha+1)_k} \left(\frac{z}{2}\right)^k$$

$$I_{U_{42}:W}^{0,\mathfrak{n}+4:V} \left(\begin{matrix} z_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_r e^{i\pi\mu_r/2} \end{matrix} \left| \begin{matrix} (1\text{-}\mu; \sigma_1, \cdots, \sigma_r), (1-2\sigma; 2\delta_1, \cdots, 2\delta_r), \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-k; \sigma_1+\rho_1, \cdots, \sigma_r+\rho_r), \end{matrix} \right. \right.$$

$$\left.\begin{matrix} (1\text{-}\lambda-k;\rho_1,\cdots,\rho_r),(1-\rho;\mu_1,\cdots,\mu_r),A:C \\ \cdots \\ \cdots \\ (1\text{-}\rho-2\sigma;\mu_1+2\delta_1,\cdots,\mu_r+2\delta_r),B:D \end{matrix}\right) \tag{5.1}$$

under the same notations and conditions that (3.1) with $\tau_i,\tau_{i^{(1)}},\cdots\tau_{i^{(r)}}\rightarrow 1$

Second integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\gamma-1}z^{\gamma+\delta}(1-x)^{\mu-1}(1-y^2)^{\delta/2-1}P_{\upsilon}^{(\alpha,\beta)}(1-zx)\,{}_2F_1[a,b;\delta;z\sqrt{1-y^2}]$$

$$I\begin{pmatrix} z_1x^{\rho_1}(1-x)^{\sigma_1}y^{\mu_1}z^{\mu_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}(1-x)^{\sigma_r}y^{\mu_r}z^{\mu_r} \end{pmatrix}\mathrm{d}x\mathrm{d}y=\frac{(\alpha+1)_{\upsilon}\Gamma(\delta)}{\upsilon!}e^{i\pi\gamma/2}\sum_{k=0}^{\upsilon}\frac{(-\upsilon)_k(1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k}\left(\frac{z}{2}\right)^k$$

$$I_{U_{33}:W}^{0,\mathfrak{n}+3:V}\left(\begin{matrix} z_1e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_re^{i\pi\mu_r/2} \end{matrix}\ \middle|\ \begin{matrix} (1\text{-}\gamma;\mu_1,\cdots,\mu_r),(1-\gamma-\delta;\mu_1,\cdots,\mu_r), \\ \cdots \\ \cdots \\ (1\text{-}\gamma-\delta+a;\mu_1,\cdots,\mu_r),(1-\gamma-\delta+b;\mu_1,\cdots,\mu_r), \end{matrix}\right.$$

$$\left.\begin{matrix} (1\text{-}\lambda-k;\rho_1,\cdots,\rho_r),A:C \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-k;\rho_1+\sigma_1,\cdots,\rho_r+\sigma_r),,B:D \end{matrix}\right) \tag{5.2}$$

under the same notations and conditions that (3.3) with $\tau_i,\tau_{i^{(1)}},\cdots\tau_{i^{(r)}}\rightarrow 1$

6. Aleph-function of two variables

If $r=2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [4] an we have

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}z^{\rho+2\sigma}(1-x)^{\mu-1}(1-y^2)^{\sigma-1}P_{\upsilon}^{(\alpha,\beta)}(1-zx)$$

$$\aleph\begin{pmatrix} z_1x^{\rho_1}(1-x)^{\sigma_1}y^{\mu_1}(1-y^2)^{\delta_1}z^{\mu_1+2\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ z_2x^{\rho_2}(1-x)^{\sigma_2}y^{\mu_2}(1-y^2)^{\delta_2}z^{\mu_2+2\delta_2} \end{pmatrix}\mathrm{d}x\mathrm{d}y=\frac{(\alpha+1)_{\upsilon}}{\upsilon!}e^{i\pi\rho/2}\sum_{k=0}^{\upsilon}\frac{(-\upsilon)_k(1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k}\left(\frac{z}{2}\right)^k$$

$$\aleph_{U_{42}:W}^{0,\mathfrak{n}+4:V}\left(\begin{array}{c} z_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_2 e^{i\pi\mu_2/2} \end{array}\right| \begin{array}{c} (1\text{-}\mu;\sigma_1,\sigma_2),(1-2\sigma;2\delta_1,2\delta_2), \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-k;\sigma_1+\rho_1,\sigma_2+\rho_2), \end{array}$$

$$\left.\begin{array}{c} (1\text{-}\lambda-k;\rho_1,\rho_2),(1-\rho;\mu_1,\mu_2),A:C \\ \cdots \\ \cdots \\ (1\text{-}\rho-2\sigma;\mu_1+2\delta_1,\mu_2+2\delta_2),B:D \end{array}\right) \tag{6.1}$$

under the same notations and conditions that (3.1) with $r=2$

Second integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\gamma-1}z^{\gamma+\delta}(1-x)^{\mu-1}(1-y^2)^{\delta/2-1}P_\upsilon^{(\alpha,\beta)}(1-zx)\,{}_2F_1[a,b;\delta;z\sqrt{1-y^2}]$$

$$\aleph\left(\begin{array}{c} z_1 x^{\rho_1}(1-x)^{\sigma_1}y^{\mu_1}z^{\mu_1} \\ \cdot \\ \cdot \\ \cdot \\ z_2 x^{\rho_2}(1-x)^{\sigma_2}y^{\mu_2}z^{\mu_2} \end{array}\right)\mathrm{d}x\mathrm{d}y = \frac{(\alpha+1)_\upsilon\Gamma(\delta)}{\upsilon!}e^{i\pi\gamma/2}\sum_{k=0}^{\upsilon}\frac{(-\upsilon)_k(1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k}\left(\frac{z}{2}\right)^k$$

$$\aleph_{U_{33}:W}^{0,\mathfrak{n}+3:V}\left(\begin{array}{c} z_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_2 e^{i\pi\mu_2/2} \end{array}\right| \begin{array}{c} (1\text{-}\gamma;\mu_1,\mu_2),(1-\gamma-\delta;\mu_1,\mu_2), \\ \cdots \\ \cdots \\ (1\text{-}\gamma-\delta+a;\mu_1,\mu_2),(1-\gamma-\delta+b;\mu_1,\mu_2), \end{array}$$

$$\left.\begin{array}{c} (1\text{-}\lambda-k;\rho_1,\rho_2),A:C \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-k;\rho_1+\sigma_1,\rho_2+\sigma_2),,B:D \end{array}\right) \tag{6.2}$$

under the same notations and conditions that (3.3) with $r=2$

7. I-function of two variables

If $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$, the Aleph-function of two variables reduces to I-function of two variables defined by Sharma et al [6] and we have.

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}z^{\rho+2\sigma}(1-x)^{\mu-1}(1-y^2)^{\sigma-1}P_\upsilon^{(\alpha,\beta)}(1-zx)$$

$$I\begin{pmatrix} z_1 x^{\rho_1}(1-x)^{\sigma_1} y^{\mu_1}(1-y^2)^{\delta_1} z^{\mu_1+2\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ z_2 x^{\rho_2}(1-x)^{\sigma_2} y^{\mu_2}(1-y^2)^{\delta_2} z^{\mu_2+2\delta_2} \end{pmatrix} \mathrm{d}x\mathrm{d}y = \frac{(\alpha+1)_\upsilon}{\upsilon!} e^{i\pi\rho/2} \sum_{k=0}^{\upsilon} \frac{(-\upsilon)_k(1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k}\left(\frac{z}{2}\right)^k$$

$$I^{0,\mathfrak{n}+4:V}_{U_{42}:W}\left(\begin{matrix} z_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_2 e^{i\pi\mu_2/2} \end{matrix}\ \middle|\ \begin{matrix} (1\text{-}\mu;\sigma_1,\sigma_2),(1-2\sigma;2\delta_1,2\delta_2), \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-k;\sigma_1+\rho_1,\sigma_2+\rho_2), \end{matrix}\right.$$

$$\left.\begin{matrix} (1\text{-}\lambda-k;\rho_1,\rho_2),(1-\rho;\mu_1,\mu_2),A:C \\ \cdots \\ \cdots \\ (1\text{-}\rho-2\sigma;\mu_1+2\delta_1,\mu_2+2\delta_2),B:D \end{matrix}\right) \tag{7.1}$$

under the same notations and conditions that (3.1) with $r=2$ and $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$.

Second integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\gamma-1}z^{\gamma+\delta}(1-x)^{\mu-1}(1-y^2)^{\delta/2-1}P_\upsilon^{(\alpha,\beta)}(1-zx)\,{}_2F_1[a,b;\delta;z\sqrt{1-y^2}]$$

$$I\begin{pmatrix} z_1 x^{\rho_1}(1-x)^{\sigma_1} y^{\mu_1} z^{\mu_1} \\ \cdot \\ \cdot \\ \cdot \\ z_2 x^{\rho_2}(1-x)^{\sigma_2} y^{\mu_2} z^{\mu_2} \end{pmatrix} \mathrm{d}x\mathrm{d}y = \frac{(\alpha+1)_\upsilon\Gamma(\delta)}{\upsilon!} e^{i\pi\gamma/2} \sum_{k=0}^{\upsilon} \frac{(-\upsilon)_k(1+\alpha+\beta+\upsilon)_k}{k!(\alpha+1)_k}\left(\frac{z}{2}\right)^k$$

$$I^{0,\mathfrak{n}+3:V}_{U_{33}:W}\left(\begin{matrix} z_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ z_2 e^{i\pi\mu_2/2} \end{matrix}\ \middle|\ \begin{matrix} (1\text{-}\gamma;\mu_1,\mu_2),(1-\gamma-\delta;\mu_1,\mu_2), \\ \cdots \\ \cdots \\ (1\text{-}\gamma-\delta+a;\mu_1,\mu_2),(1-\gamma-\delta+b;\mu_1,\mu_2), \end{matrix}\right.$$

$$\left.\begin{matrix} (1\text{-}\lambda-k;\rho_1,\rho_2),A:C \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-k;\rho_1+\sigma_1,\rho_2+\sigma_2),,B:D \end{matrix}\right) \tag{7.2}$$

under the same notations and conditions that (3.3) with $r=2$ and $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$.

8. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function ,multivariable Fox's H-function, Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results

References

[1] Erdelyi A et al. Tables of integral transforms. Vol II. McGraw-Hill, New-York (1954).

[2] Mac-Robert T.M. Beta function formulae and integrals involving E-function. Math.Ann. 142, page 450-452 (1961).

[3] Mathai A.M. And Saxena R.K. Generalized hypergeometric functions with applications in statistics and physical sciences. Lecture Notes in Mathematics. Vol 348. Springer-Verlag. New York (1973).

[4] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page1-13.

[5] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[6] C.K. Sharma and P.L. mishra : On the I-function of two variables and its properties. Acta Ciencia Indica Math , 1991 Vol 17 page 667-672.

Personal adress : 411 Avenue Joseph Raynaud
Le parc Fleuri , Bat B
83140 , Six-Fours les plages
Tel : 06-83-12-49-68
Department : VAR
Country : FRANCE