Roman Labeling of graphs and Application to Military Strategy

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Abstract:
In this paper, we introduce a new type of graph labeling called Roman labeling and a graph parameter called Roman number. Its properties are studied and its values for special types of graphs are explored.

Keyword: *Graph, labeling*

1. INTRODUCTION

The majority of graph theory research on graph labeling pays attention only to finding vertex labeling function that satisfies some specified property for the induced edge labeling, paying little attention to the graph structure [6]. In his Doctoral Dissertation [2], Jason Robert Lewis suggested several new or little-studied graph parameters. Several studies were made in applying such parameters to Roman defense strategy [2, 3, 4, 5].

The idea behind it is that if we consider the edges and the vertices of a graph as some streets and the junctions, which are the meeting points of the streets, then the label of a vertex is the number of soldiers deployed at that junction and we require that every street (edge) should be guarded by at least 1 soldier. That is, in case of any street having no soldiers, then there should be an adjacent junction with two soldiers so that one of them can be deployed to the former junction in case of emergency.

These works motivated us to define a new type of graph labeling, namely, Roman Labeling. We study the properties of Roman Labeling and the values of Roman number for special types of graphs. For the terms and definitions not explicitly here, refer Harary [7]

2. *Main Results*

In this section, we consider only connected graphs. The disconnected graphs can be studied though their components.

Let G be a connected graph. Roman labeling of G is a function f: V(G) → {0, 1, 2} such that any vertex with label 0 must be adjacent to a vertex with label 2. The function value f(v) of a vertex v of the graph G is called the label of v.

It can be easily seen that if G has a Roman labeling, then for any edge e = {u, v}, either both u and v are adjacent to vertices with labels at least 1 or the edge e is incident with a vertex with label 2.

Clearly, the function, f, partitions the vertex set, V(G) into three vertex subsets, V_0, V_1 and V_2 .which are the subsets of V(G) with labels 0, 1,2 respectively.

Weight of a Roman labeling, f is defined as the sum of all vertex labels. Roman number of a graph G is defined as the minimum weight of a Roman labeling on G and is denoted by R(G). A Roman labeling with the minimum weight is called a Roman Number.

We observe that a vertex, u is either an end-vertex of an edge, e = xy or is incident to an edge e = xy. Based on this, we define the edge neighborhood of a vertex v as follows:

$$E(v) = \{e \in E(G): v \textit{ is an endvertex of } e\}$$
$$= \{e \in E(G): e \textit{ is incident with } v\}$$

Theorem 2.1. Let f be a minimal Roman labeling function of a graph G. Then there is no edge connecting V_1 and V_2.

Proof.If possible, suppose that there is an edge e = uv, such that $u \in V_1$ and $v \in V_2$.. Since v is adjacent to u, it is incident to all edges incident at u. So we can reduce f(u) from one tozero, which is a contradiction to the fact that f be a minimal Roman labeling function of a graph G

Theorem 2.2. For a graph G, R(G) = 1 if and only if $E(v) = E(G)$ for some vertex v of G.

Proof. If R(G) = 1 , then all edges of G are adjacent to v. Hence, $E(v) = E(G)$. Conversely, if $E(v) = E(G)$, then v is incident with all the edges of G, then the function f(v) = 1 and f(x) = 0 for all other vertices of G is a Roman labeling function with R(G) = 1.

Theorem 2.3. For a graph G, R(G)= 1 if and only if G is a star graph.

Proof. First suppose R(G) = 1. Then there is no vertex with label 2 and exactly one vertex, v, with label 1. Then v is incident with all the edges of G. Then, G is a star graph. Conversely, suppose G is a star graph. Then, the function defined by f(v) = 1 and f(u) = 0 for any other vertex u, is a Roman labeling of G with R(G) = 1.

Theorem 2.4. If R(G)= 2, then $E(v) = E(G)$, for some vertex v of G.

Proof. If R(G)= 2 and f is a Roman labeling of G, then we have 2 cases to consider.

Case-1: f(v) = 2 for exactly one vertex v. Then all vertices with label 0 must be adjacent to v and since S(G)=2, they constitutes the entire set of vertices, so that $E(v) = E(G)$

Case: 2: There are exactly two vertices, say u, v with f(u)=f(v)=1. Then, all other vertices with label 0 and no vertex must have label 2. Hence G is precisely the edge {u, v}, so that $E(G) = E(v)$.

Theorem 2.5. If G is not a star graph and the minimum eccentricity and the maximum eccentricity of vertices in G are 1 and 2 respectively, then R(G)= 2.

Proof. Since G is not a star graph and minimum eccentricity is 1, there exists a vertex v, which is adjacent to all other vertices. So the function defined by f(v) = 2 and f(u) = 0 for any other vertex u, is a Roman labeling so that R(G)=2.

Theorem 2.6. If R(G) = 3, then there exists a Roman labeling, f and exactly two vertices u, v such that f(u) = 1and f(v) = 2.

Proof. Let f be a minimal Roman labeling of the given graph G. Then R(G)= 3 is possible in the following two ways:

Case-1: There exist exactly two vertices u, v such that f(u) = 1and f(v) = 2. This is the required condition in the theorem.

Case: 2 There exist three vertices u, v, w such that f(u)=f(v)=f(w)=1. Then we can find another vertex x and a vertex among u, v and w, sayu such that the new function g defined by g(x) = 2. g(u) = 1 and g(y) = 0 for all the remaining vertices is a a Roman labeling of G. Since G is connected, there exist a pair of vertices among u, v, w, say p and q, such that the induced subgraphs induced by the neighborhoods, N[p] andN[q] have one vertex in common. We can define a new Roman labeling by assigning 2 to this common vertex, 1 to the third vertex and 0 to the remaining vertices. Thus this case reduces to case-1 and hence the theorem.

Theorem2.7. If G = Kn, n>2, then R(G)= 2

Proof. In G = K_n, $E(v) = E(G)$ and $E(v) \neq E(G)$ for all vertices of G, the result follows.

Theorem2.8. If G = P_n, then R(G)= 2r or 2r-1 according as if n = 5r, 5r – 1 or n = 5r − 2, 5r – 3, 5r − 4

Proof. Along the path we take a block of four edges. Let v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5 be the vertices and edges on the path. Arranging two guards at the vertex v_3 we can guard all the edges in the block. We can do the same for each block five vertices and four edges. We cannot guard a block by less number of guards. Thus for a path which contains n = 5r vertices 2r guards are needed.

For a path containing 5r − 2, 5r − 3 or 5r − 4 vertices, there are r – 1 complete blocks of five vertices. We require 2(r − 1) guards to take care of the edges in the blocks. After taking the blocks from one end if there remains either one or two vertices at the other end, one guard arranged at the first vertex on the last section can protect the last edges. If there are three vertices at the last portion, say u_1, u_2, u_3, one guard is needed at u_3. In all these cases $R(P_n) = 2r − 1$. If in the remaining block there are four vertices, then two guards are to be placed at the vertex u_4. Hence $R(P_n)$ = 2r.

Theorem 2.9. If G = C_n, then R(G)=2r or 2r–1 according as n=5r,5r–1 or n=5r−2, 5r–3, 5r−4

Proof. An argument similar to that used in the previous proof is sufficient to prove the result. Let the graph G be decomposed into sub-graphs {G_i}, such that each subgraph is any one of the following type,

• Either the minimum eccentricity of vertices in G_i is 2 and the maximum eccentricity is 3 or 4

• Minimum eccentricity and maximum eccentricity of vertices in G_i are respectively 1 and 2.

We denote the decomposition of G by D = {G_i}. Some of these graphs have $R(G_i)$ = 2 and the remaining have $R(G_i)$ = 1. We define a related number N(D) = a+2b, where a and b are the number of sub-graphs having $R(G_i)$ =1 and 2 respectively.

As illustrated below, a graph has many decompositions. The graph G given in the Figure 2.1 has the vertex set V = {v_1, v_2, ..., v_9}.It is decomposed into three subgraphs G_1, G_2 and G_3, which are given in Figure 2.2 and another decomposition into subgraphs H_1, H_2 are given in Figure 2.3.

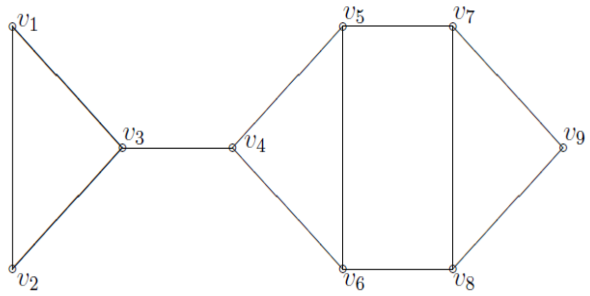

Figure 2.1

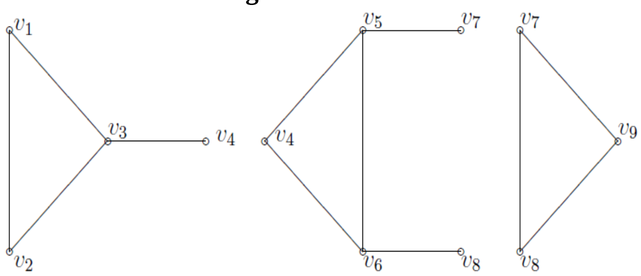

Figure 2.2

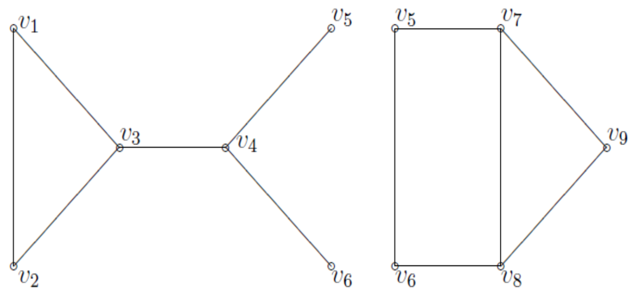

Figure 2.3

Theorem 2.10. For complete r-partite graph, $G = K_{m_1, m_2, \ldots, m_r}$ R(G)=2 where $r \geq 2$

Proof. Let V = $V_1 \cup V_2 \cup V_3 \ldots . V_r$ be the partition of the vertex set of V(G). Take a vertex v from V_1. Every edge in G is either incident with v or adjacent to an edge, which is incident with v. So the function f(v) = 2 and f(u) = 0 for any other vertex u is a Roman labeling of G.

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