On multiple integral transformations with the multivariable I-functions

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ABSTRACT
In this paper, two multiple integral transformations of the multivariable I-function defined by Prathima and Nambisan [3] with general arguments have been established. Next two new multiple integrals for the multivariable I-function have been evaluated by employing the transformations. The integrals have further been generalized to give two another multiple integrals involving product of two multivariable I-functions in their integrands.

Keywords : Serie expansion of the multivariable I-function, multivariable I-function, multiple integral.

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1. Introduction

The multivariable I-function defined by Prathima and Nambisan [3] is an extension of the multivariable H-function defined by Srivastava and Panda [4,5]. It is defined in term of multiple Mellin-Barnes type integral :

$$\bar{I}(z_1,\cdots,z_r) = I^{0,N:M_1,N_1;\cdots;M_r,N_r}_{P,Q:P_1,Q_1;\cdots;P_r,Q_r}\left(\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,P} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,Q} : \end{array}\right.$$

$$\left.\begin{array}{c} (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,N_1},(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{N_1+1,P_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,N_r},(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{N_r+1,P_r} \\ (d_j^{(1)},\delta_j^{(1)};1)_{1,M_1},(d_j^{(1)},\delta_j^{(1)};D_1)_{M_1+1,Q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{1,M_r},(d_j^{(r)},\delta_j^{(r)};D_r)_{M_r+1,Q_r} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi(s_1,\cdots,s_r)$, $\theta_i(s_i)$ for $i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{Q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.3}$$

$$\phi_i(s_i) = \frac{\prod_{j=1}^{N_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{M_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=N_i+1}^{P_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=M_i+1}^{Q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \tag{1.4}$$

For more details, see Prathima and Nambisan [3].

We can obtain the series representation and behaviour for small values for the function $I(z_1,\cdots,z_r)$ defined and represented by (1.1). The series representation may be given as follows , which is valid under the following conditions :

$$\delta_i^{(h)}[d_i^{(j)}+r] \neq \delta_i^{(j)}[d_i^{(h)}+\mu] \text{ for } j \neq h, j,h=1,\cdots,M_i, r,\mu=0,1,2,\cdots$$

$$U_i=\sum_{j=1}^{P} A_j\alpha_j^{(i)}-\sum_{j=1}^{Q} B_j\beta_j^{(i)}+\sum_{j=1}^{P_i} C_j^{(i)}\gamma_j^{(i)}-\sum_{j=M_i+1}^{Q_i} D_j^{(i)}\delta_j^{(i)} \leqslant 0, i=1,\cdots,r \quad \text{and } z_i \neq 0$$

and if all the poles of (1.11) are simple ,then the integral (1.11) can be evaluated with the help of the Residue theorem to give

$$\bar{I}(z_1,\cdots,z_r)=\sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty}\phi\frac{\prod_{i=1}^r \phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r \delta_{G^{(i)}}^{(i)}\prod_{i=1}^r g_i!} \tag{1.5}$$

where ϕ_1 and ϕ_i are defined by

$$\phi=\frac{\prod_{j=1}^{N}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^r \alpha_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum_{i=1}^r \alpha_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{Q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^r \beta_j^{(i)}\eta_{G_i,g_i}\right)} \tag{1.6}$$

and

$$\phi_i=\frac{\prod_{j=1}^{N_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{M_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=N_i+1}^{P_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=M_i+1}^{Q_i}\Gamma^{(D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}\eta_{G_i,g_i}\right)}, i=1,\cdots,r \tag{1.7}$$

where

$$\eta_{G_i,g_i}=\frac{d_{G^{(i)}}^{(i)}+g_i}{\delta_{G^{(i)}}^{(i)}}, i=1,\cdots,r \tag{1.8}$$

and

$$I(z_1,\cdots,z_s) = I_{\mathbf{p},\mathbf{q}:p_1,q_1;\cdots;p_s,q_s}^{0,\mathbf{n}:m_1,n_1;\cdots;m_s,n_s}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(s)};A_j)_{1,\mathbf{p}}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(s)};B_j)_{1,\mathbf{q}}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(s)},\gamma_j^{(s)};C_j^{(s)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(s)},\delta_j^{(s)};D_j^{(s)})_{1,q_r} \end{array}\right) \tag{1.9}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\phi'(u_1,\cdots,u_s)\prod_{i=1}^{s}\zeta_i(u_i)Z_i^{u_i}\mathrm{d}u_1\cdots\mathrm{d}u_s \tag{1.10}$$

where $\phi(u_1,\cdots,u_s), \zeta_i(u_i)$ for $i=1,\cdots,s$ are given by :

$$\phi'(u_1,\cdots,u_s)=\frac{\prod_{j=1}^{\mathbf{n}}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^s \alpha_j^{(i)}u_j\right)}{\prod_{j=\mathbf{n}+1}^{\mathbf{p}}\Gamma^{A_j}\left(a_j-\sum_{i=1}^s \alpha_j^{(i)}u_j\right)\prod_{j=\mathbf{m}+1}^{\mathbf{q}}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^s \beta_j^{(i)}u_j\right)} \tag{1.11}$$

$$\zeta_i(u_i) = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}u_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}u_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}u_i\right)\prod_{j=m_i+1}^{q_i'}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}u_i\right)} \tag{1.12}$$

For more details, see Prathima and Nambisan [3]. Following the result of Braaksma the I-function of r variables is analytic if

$$U_i = \sum_{j=1}^{\mathbf{p}} A_j\alpha_j^{(i)} - \sum_{j=1}^{\mathbf{q}} B_j\beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)}\delta_j^{(i)} \leqslant 0, i = 1,\cdots,s \tag{1.13}$$

The integral (2.1) converges absolutely if

where$|arg(z_k)| < \frac{1}{2}\Delta_k\pi, k = 1,\cdots,s$

$$\Delta_k = -\sum_{j=\mathbf{n}+1}^{\mathbf{p}} A_j\alpha_j^{(k)} - \sum_{j=1}^{\mathbf{q}} B_j\beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)}\delta_j'^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)}\gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)}\gamma_j^{(k)} > 0 \tag{1.14}$$

In this paper, we shall note.

$$X = m_1, n_1;\cdots;m_s, n_s \quad : Y = p_1, q_1;\cdots;p_s, q_s \tag{1.15}$$

$$\mathbb{A} = (a_j; A_j^{(1)},\cdots,A_j^{(s)}; A_j)_{1,\mathbf{p}} : (\mathrm{c}_j^{(1)},\gamma_j^{(1)}; C_j^{(1)})_{1,p_1};\cdots;(c_j^{(s)},\gamma_j^{(s)}; C_j^{(s)})_{1,p_s} \tag{1.16}$$

$$\mathbb{B} = (b_j; B_j^{(1)},\cdots,B_j^{(s)}; B_j)_{1,\mathbf{q}} :(\mathrm{d}_j^{(1)},\delta_j^{(1)}; D_j^{(1)})_{1,q_1};\cdots;(d_j^{(s)},\delta_j^{(s)}; D_j^{(s)})_{1,q_s} \tag{1.17}$$

2. Multiple integral transformations

In this section, following multiple transformations are established.

$$\int_0^\infty\cdots\int_0^\infty \prod_{j=1}^p x_j^{s_j-1}\; I\left(\begin{array}{c} \mathrm{Z}_1\left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^p x_j^{v_j^{(1)}} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{Z}_s\left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^p x_j^{v_j^{(s)}} \end{array}\right)$$

$$f\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)\mathrm{d}x_1\cdots\mathrm{d}x_p = \prod_{j=1}^p \frac{(c_j'')^{-s_j/t_j}}{t_j}\int_0^\infty z^{S-1} f(z)$$

$$I_{\mathbf{p}+p,\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n}+p;X}\left(\begin{array}{c} \mathrm{Z}_1\prod_{j=1}^p (c_j'')^{-v_j^{(1)}/t_j} z^{V_1+\mu_1}(1+z)^{v_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s\prod_{j=1}^p (c_j'')^{-v_j^{(s)}/t_j} z^{V_s+\mu_s}(1+z)^{v_s} \end{array}\left|\begin{array}{c} \left(1-\frac{s_j}{t_j} : \frac{v_j^{(1)}}{t_j},\cdots,\frac{v_j^{(s)}}{t_j}; 1\right)_{1,p}, \mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-S: } \mathrm{V}_1,\cdots,V_s; 1), \mathbb{B} \end{array}\right.\right)\mathrm{d}z \tag{2.1}$$

where $S = \sum_{j=1}^p \frac{s_j}{t_j}, V_i = \sum_{j=1}^p \frac{v_j^{(i)}}{t_j}; i = 1,\cdots,s$

provided that $\mu_i, v_j^{(i)} \geqslant 0; j = 1, \cdots, p, i = 1, \cdots, s\,; v_i > 0; i = 1, \cdots, s$

$min\{c''_j, t_j, Re(s_j)\} > 0$ for $j = 1, \cdots, p$ and $|arg(Z_k)| < \frac{1}{2}\Delta_k\pi, k = 1, \cdots, s$

$v_i - \mu_i - V_i > 0, i = 1, \cdots, s$ and the function f is so prescribed that the various integrals (2.1) exist, also,

$$\int\cdots\int_{R_p}\prod_{j=1}^{p} x_j^{s_j-1}\ I\begin{pmatrix} Z_1\left(1-\sum_{j=1}^{p} c''_j x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^{p} c'_j x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^{p} x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ Z_s\left(1-\sum_{j=1}^{p} c''_j x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^{p} c'_j x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^{p} x_j^{v_j^{(s)}} \end{pmatrix}$$

$$f\left(\sum_{j=1}^{p} c''_j x_j^{t_j}\right)\mathrm{d}x_1\cdots\mathrm{d}x_p = \prod_{j=1}^{p}\frac{(c''_j)^{-s_j/t_j}}{t_j}\int_0^1 z^{S-1} f(z)$$

$$I^{\mathbf{m},\mathbf{n}+p;X}_{\mathbf{p}+p,\mathbf{q}+1;Y}\left(\begin{matrix} Z_1\prod_{j=1}^{p}(c''_j)^{-v_j^{(1)}/t_j} z^{V_1+\mu_1}(1-z)^{v_1} \\ \cdot \\ \cdot \\ \cdot \\ Z_s\prod_{j=1}^{p}(c''_j)^{-v_j^{(s)}/t_j} z^{V_s+\mu_s}(1-z)^{v_s} \end{matrix}\ \middle|\ \begin{matrix} \left(1-\frac{s_j}{t_j}:\frac{v_j^{(1)}}{t_j},\cdots,\frac{v_j^{(s)}}{t_j};1\right)_{1,p},\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-S}: \mathrm{V}_1,\cdots,V_s;1),\mathbb{B} \end{matrix}\right)\mathrm{d}z \tag{2.2}$$

where $S = \sum_{j=1}^{p}\frac{s_j}{t_j}, V_i = \sum_{j=1}^{p}\frac{v_j^{(i)}}{t_j}; i = 1, \cdots, s$

provided that $v_i, \mu_i, v_j^{(i)} \geqslant 0; j = 1, \cdots, p, i = 1, \cdots, s$

$min\{c''_j, t_j, Re(s_j)\} > 0$ for $j = 1, \cdots, p$ and $|arg(Z_k)| < \frac{1}{2}\Delta_k\pi, k = 1, \cdots, s$

and the function f is so prescribed that the various integrals in (2.2) exist. Also R_p is the region defined by

$x_j \geqslant 0$ and $\sum_{i=1}^{p} c''_j x^{t_j} \leqslant 1 (j = 1, \cdots, p)$

Proof of (2.1)

Writing contour integral for the multivariable I-function in the l.h.s. of (2.1), changing the order of integrations, which is justified under the conditions mentioned with (2.1), we find that

$$l.h.s. = \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi'(u_1,\cdots,u_s)\prod_{i=1}^{s}\zeta_i(u_i)Z_i^{u_i}\left\{\int_0^\infty\cdots\int_0^\infty\right.$$

$$\left.\left(\sum_{j=1}^{p} c''_j x_j^{t_j}\right)^{\sum_{i=1}^{s}\mu_i u_i}\left(1+\sum_{j=1}^{p} c''_j x_j^{t_j}\right)^{-\sum_{i=1}^{s} v_i u_i} f\left(\sum_{j=1}^{p} c''_j x_j^{t_j}\right)\mathrm{d}x_1\cdots\mathrm{d}x_p\right\}\mathrm{d}u_1\cdots\mathrm{d}u_s \tag{2.3}$$

Now, by an appeal of the following useful result of Edwards [1, page 172] :

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^p x_j^{s_j-1} f\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_p = \frac{\prod_{j=1}^p \left(\frac{s_j}{t_j}\right)}{\Gamma\left(\sum_{j=1}^p \left(\frac{s_j}{t_j}\right)\right)} \prod_{j=1}^p \frac{(c_j'')^{-s_j/t_j}}{t_j} \int_0^\infty z^{S-1} f(z) \mathrm{d}z \tag{2.4}$$

where $S = \sum_{j=1}^p \frac{s_j}{t_j}; i = 1, \cdots, s$ and $min\{c_j'', t_j, Re(s_j)\} > 0$ for $j = 1, \cdots, p$

and using (1.10) again, we get the right hand side of (2.1).

To prove (2.2), we follow a method similar to that given above.

3. Multiple integrals

In the early section, we have remarked that a number of multiple integrals can be easily obtained by choosing the function f suitably. For the sake of illustration, we evaluate here following two simple and interesting multiple integrals :

First integral

Taking $f(z) = z^\sigma (1+z)^{-\lambda}$ in (2.1), using (1.10) and a known result [2, page 10], to evaluate the $z-$integral, we arrive at the following interesting and new integral.

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^p x_j^{s_j-1} \; I\begin{pmatrix} \mathrm{Z}_1 \left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_1} \left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_1} \prod_{j=1}^p x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ \mathrm{Z}_s \left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_s} \left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_s} \prod_{j=1}^p x_j^{v_j^{(s)}} \end{pmatrix}$$

$$\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^\sigma \left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-\lambda} \mathrm{d}x_1 \cdots \mathrm{d}x_p = \prod_{j=1}^p \frac{(c_j'')^{-s_j/t_j}}{t_j}$$

$$I^{\mathbf{m},\mathbf{n}+p+2;X}_{\mathbf{p}+p+2,\mathbf{q}+2;Y}\left(\begin{array}{c|c} \mathrm{Z}_1 \prod_{j=1}^p (c_j'')^{-v_j^{(1)}/t_j} & \left(1-\frac{s_j}{t_j} : \frac{v_j^{(1)}}{t_j}, \cdots, \frac{v_j^{(s)}}{t_j}; 1\right)_{1,p}, \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_s \prod_{j=1}^p (c_j'')^{-v_j^{(s)}/t_j} & \text{(1-S: } \mathrm{V}_1, \cdots, V_s; 1), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\sigma - S : \mu_1 + V_1, \cdots, \mu_s + V_s), (1+\sigma-\lambda+S : v_1 - \mu_1 - V_1, \cdots, v_s - \mu_s - V_s; 1), \mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\lambda; v_1, \cdots, v_s; 1), \mathbb{B} \end{array}\right) \tag{3.1}$$

where $S = \sum_{j=1}^p \frac{s_j}{t_j}, V_i = \sum_{j=1}^p \frac{v_j^{(i)}}{t_j}; i = 1, \cdots, s$

provided that $\mu_i, v_j^{(i)} \geqslant 0; j = 1, \cdots, p, i = 1, \cdots, s \,; v_i > 0; i = 1, \cdots, s$

$min\{c''_j, t_j, Re(s_j), Re(\sigma+1), Re(\lambda)\} > 0$ for $j = 1, \cdots, p$

$v_i - \mu_i - V_i > 0, i = 1, \cdots, s$ and $|arg(Z_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, s$

$$0 < Re(S+\sigma) + \sum_{i=1}^{s}(V_i+\mu_i) \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) < Re(\lambda) + \sum_{i=1}^{s} v_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)$$

Second integral

Taking $f(z) = z^{\sigma}(1-z)^{-\lambda}$ in (2.2), using (1.10) and a known result [2, page 10], to evaluate the $z-$integral, we arrive at the following interesting and new integral.

$$\int\cdots\int_{R_p}\prod_{j=1}^{p} x_j^{s_j-1} I\left(\begin{array}{c} Z_1\left(1-\sum_{j=1}^p c''_j x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^p c''_j x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^p x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ Z_s\left(1-\sum_{j=1}^p c''_j x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^p c''_j x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^p x_j^{v_j^{(s)}} \end{array}\right)$$

$$\left(\sum_{j=1}^{p} c''_j x_j^{t_j}\right)^{\sigma}\left(1-\sum_{j=1}^{p} c''_j x_j^{t_j}\right)^{-\lambda} \mathrm{d}x_1\cdots\mathrm{d}x_p = \prod_{j=1}^{p}\frac{(c''_j)^{-s_j/t_j}}{t_j}$$

$$I^{\mathbf{m},\mathbf{n}+p+2;X}_{\mathbf{p}+p+2,\mathbf{q}+2;Y}\left(\begin{array}{c} Z_1\prod_{j=1}^p (c''_j)^{-v_j^{(1)}/t_j} \\ . \\ . \\ . \\ Z_s\prod_{j=1}^p (c''_j)^{-v_j^{(s)}/t_j} \end{array}\left|\begin{array}{c} \left(1-\frac{s_j}{t_j}:\frac{v_j^{(1)}}{t_j},\cdots,\frac{v_j^{(s)}}{t_j};1\right)_{1,p}, \\ . \\ . \\ (1\text{-}S: V_1,\cdots,V_s;1), \end{array}\right.\right.$$

$$\left.\begin{array}{c} (1\text{-}\sigma-S;\mu_1+V_1,\cdots,\mu_s+V_s;1),(-\lambda+S:v_1,\cdots,v_s;1),\mathbb{A} \\ . \\ . \\ (\text{-}\lambda-\sigma-S;\mu_1+v_1+V_1,\cdots,\mu_s+v_s+V_s;1),\mathbb{B} \end{array}\right) \tag{3.2}$$

where $S = \sum_{j=1}^{p}\frac{s_j}{t_j}, V_i = \sum_{j=1}^{p}\frac{v_j^{(i)}}{t_j}; i = 1,\cdots,s$, also R_p is the region defined by

$x_j \geqslant 0$ and $\sum_{i=1}^{p} c''_j x^{t_j} \leqslant 1 (j = 1, \cdots, p)$

provided that $v_i, \mu_i, v_j^{(i)} \geqslant 0; j = 1, \cdots, p, i = 1, \cdots, s$

$min\{c''_j, t_j, Re(s_j), Re(\sigma+1), Re(\lambda+1)\} > 0$ for $j = 1, \cdots, p$

$v_i - \mu_i - V_i > 0, i = 1, \cdots, s$ and $|arg(Z_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, s$

$$0 < Re(S+\sigma) + \sum_{i=1}^{s}(V_i+\mu_i) \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) \text{ and } Re(\lambda+1) + \sum_{i=1}^{s} v_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

4. Generalized of the integrals (3.1) and (3.2)

First integral

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^p x_j^{s_j-1} \quad \left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^\sigma \left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-\lambda}$$

$$\bar{I}\left(\begin{array}{c} z_1\left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-u}\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\rho}\prod_{j=1}^p x_j^{u_j^{(1)}} \\ \cdots \\ \cdots \\ z_r\left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-u}\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\rho}\prod_{j=1}^p x_j^{u_j^{(s)}} \end{array}\right)$$

$$I\left(\begin{array}{c} Z_1\left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^p x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ Z_s\left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^p x_j^{v_j^{(s)}} \end{array}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_p$$

$$= \sum_{G_i=1}^{M_i}\sum_{g_i=1}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r \delta_{G^{(i)}}^{(i)} \prod_{i=1}^r g_i!} \prod_{j=1}^p \frac{(c_j'')^{-(s_j+u_j^{(i)}\eta_{G_i,g_i})/t_j}}{t_j}$$

$$I^{\mathbf{m},\mathbf{n}+p+2;X}_{\mathbf{p}+p+2,\mathbf{q}+2;Y}\left(\begin{array}{c} Z_1\prod_{j=1}^p (c_j'')^{-v_j^{(1)}/t_j} \\ \cdot \\ \cdot \\ \cdot \\ Z_s\prod_{j=1}^p (c_j'')^{-v_j^{(s)}/t_j} \end{array}\middle| \begin{array}{c} \left(1-\frac{s_j+u_j^{(i)}\eta_{G_i,g_i}}{t_j} : \frac{v_j^{(1)}}{t_j},\cdots,\frac{v_j^{(s)}}{t_j};1\right)_{1,p}, \\ \cdot \\ \cdot \\ (1\text{-}S(\eta_{G_i,g_i}) : V_1,\cdots,V_s;1), \end{array}\right.$$

$$(1\text{-}\sigma-\rho\eta_{G_i,g_i}-S(\eta_{G_i,g_i});\mu_1+V_1,\cdots,\mu_s+V_s;1),$$
$$\cdot$$
$$\cdot$$
$$\cdots$$

$$\left.\begin{array}{c}(1+\sigma+\rho\eta_{G_i,g_i}-\lambda-\mu\eta_{G_i,g_i}+S(\eta_{G_i,g_i}) : v_1-\mu_1-V_1,\cdots,v_s-\mu_s-V_s;1),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\lambda-u\eta_{G_i,g_i} : v_1,\cdots,v_s;1),\mathbb{B}\end{array}\right) \tag{4.1}$$

where $S(\eta_{G_i,g_i}) = \sum_{j=1}^p \frac{s_j+u_j^{(i)}\eta_{G_i,g_i}}{t_j}$, ϕ, ϕ_i and η_{G_i,g_i} are defined respectively by (1.6), (1.7) and (1.8).

The integral (4.1) is valid if :

(i) the sets of conditions given with (3.1) are satisfied.

(ii) $u = \rho - \sum_{j=1}^{p} \frac{u_j^{(i)}}{t_j} > 0, u, \rho, u_j^{(i)} \geqslant 0 (j = 1, \cdots, p; i = 1, \cdots, r)$

(iii) The multiple series on the right hand side are absolutely convergents.

(iv) $U_i = \sum_{j=1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{P_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=M_i+1}^{Q_i} D_j^{(i)} \delta_j^{(i)} < 0, i = 1, \cdots, r$

Second integral

$$\int \cdots \int_{R_p} \prod_{j=1}^{p} x_j^{s_j - 1} \left(\sum_{j=1}^{p} c_j'' x_j^{t_j} \right)^{\sigma} \left(1 - \sum_{j=1}^{p} c_j'' x_j^{t_j} \right)^{-\lambda}$$

$$\bar{I} \left(\begin{array}{c} z_1 \left(1 - \sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{-u} \left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{\rho} \prod_{j=1}^{p} x_j^{u_j^{(1)}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ z_r \left(1 - \sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{-u} \left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{\rho} \prod_{j=1}^{p} x_j^{u_j^{(s)}} \end{array} \right)$$

$$I \left(\begin{array}{c} Z_1 \left(1 - \sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{-v_1} \left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{\mu_1} \prod_{j=1}^{p} x_j^{v_j^{(1)}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ Z_s \left(1 - \sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{-v_s} \left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{\mu_s} \prod_{j=1}^{p} x_j^{v_j^{(s)}} \end{array} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_p$$

$$= \sum_{G_i=1}^{M_i} \sum_{g_i=1}^{\infty} \phi \frac{\prod_{i=1}^{r} \phi_i z_i^{\eta_{G_i,g_i}} (-)^{\sum_{i=1}^{r} g_i}}{\prod_{i=1}^{r} \delta_{G^{(i)}}^{(i)} \prod_{i=1}^{r} g_i!} \prod_{j=1}^{p} \frac{(c_j'')^{-(s_j + u_j^{(i)} \eta_{G_i,g_i})/t_j}}{t_j}$$

$$I_{\mathbf{p}+p+2, \mathbf{q}+2; Y}^{\mathbf{m}, \mathbf{n}+p+2; X} \left(\begin{array}{c} Z_1 \prod_{j=1}^{p} (c_j'')^{-v_j^{(1)}/t_j} \\ \cdot \\ \cdot \\ \cdot \\ Z_s \prod_{j=1}^{p} (c_j'')^{-v_j^{(s)}/t_j} \end{array} \left| \begin{array}{c} \left(1 - \frac{s_j + u_j^{(i)} \eta_{G_i,g_i}}{t_j} : \frac{v_j^{(1)}}{t_j}, \cdots, \frac{v_j^{(s)}}{t_j}; 1\right)_{1,p}, \\ \cdot \\ \cdot \\ (1\text{-}S(\eta_{G_i,g_i}) : V_1, \cdots, V_s; 1), \end{array} \right. \right.$$

$$\left. \begin{array}{c} (1\text{-}\sigma - \rho\eta_{G_i,g_i} - S(\eta_{G_i,g_i}); \mu_1 + V_1, \cdots, \mu_s + V_s; 1), (-\lambda - u\eta_{G_i,g_i} : v_1, \cdots, v_s; 1), \mathbb{A} \\ \cdot \\ \cdot \\ (\text{-}\lambda - \sigma - (u+\rho)\eta_{G_i,g_i} - S(\eta_{G_i,g_i}) : \mu_1 + v_1 + V_1, \cdots, \mu_s + v_s + V_s; 1), \mathbb{B} \end{array} \right) \quad (4.2)$$

where $S(\eta_{G_i,g_i}) = \sum_{j=1}^{p} \frac{s_j + u_j^{(i)} \eta_{G_i,g_i}}{t_j}$, ϕ, ϕ_i and η_{G_i,g_i} are defined respectively by (1.6), (1.7) and (1.8).

also R_p is the region defined by $x_j \geqslant 0$ and $\sum_{i=1}^{p} c_j'' x^{t_j} \leqslant 1 (j = 1, \cdots, p)$

The integral (4.2) is valid if :

(i) the sets of conditions given with (3.2) are satisfied.

(ii) $u, \rho, u_j^{(i)} \geqslant 0 (j = 1, \cdots, p; i = 1, \cdots, r)$

(iii) The multiple series on the right hand side are absolutely convergents.

(iv) $U_i = \sum_{j=1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{P_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=M_i+1}^{Q_i} D_j^{(i)} \delta_j^{(i)} < 0, i = 1, \cdots, r$

Proof

To prove (4.1), first expressing the multivariable I-function defined by Prathima and Nambisan [3] in series with the help of (1.1) and we interchange the order of summations and $(x_1 \cdots x_p)-$integral (which is permissible under the conditions stated). Now collect the power of $\sum_{j=1}^{p} c_j'' x_j^{t_j}$ and $1 - \sum_{j=1}^{p} c_j'' x_j^{t_j}$ and use the integral (3.1), we obtain the desired result.

The integral (4.2) can be proved in a similar manner.

5. Conclusion

The integral (4.1) and (4.2) involve the multivariable I-functions defined by Prathima and Nambisan [3], which is quite general in character. On specializing the various parameters of these functions, a number of new obtained double, triple and multiple integrals can be obtained involving special functions of one and several variables.

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