On multiple integral transformations with the multivariable A-functions

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ABSTRACT
In this paper, two multiple integral transformations of the multivariable A-function defined by Gautam and Asgar [3] with general arguments have been established. Next two new multiple integrals for the multivariable A-function have been evaluated by employing the transformations. The integrals have further been generalized to give two another multiple integrals involving product of two multivariable A-functions in their integrands.

Keywords : Serie expansion of the multivariable A-function, multivariable A-function, multiple integral.

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1.Introduction

The serie representation of the multivariable A-function is given by Gautam and Asgar [3] as

$$A[u_1,\cdots,u_r]=A^{0,\lambda:(\alpha',\beta');\cdots;(\alpha^{(r)},\beta^{(r)})}_{A,C:(M',N');\cdots;(M^{(r)},N^{(r)})}\left(\begin{array}{c}\mathrm{u}_1\\ \cdot\\ \cdot\\ \mathrm{u}_r\end{array}\middle|\begin{array}{c}[(\mathrm{g}_j);\gamma',\cdots,\gamma^{(r)}]_{1,A}:\\ \cdots\\ [(\mathrm{f}_j);\xi',\cdots,\xi^{(r)}]_{1,C}:\end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{q}^{(1)},\eta^{(1)})_{1,M^{(1)}};\cdots;(q^{(r)},\eta^{(r)})_{1,M^{(r)}}\\ \cdots\\ \cdots\\ (\mathrm{p}^{(1)},\epsilon^{(1)})_{1,N^{(1)}};\cdots;(p^{(r)},\epsilon^{(r)})_{1,N^{(r)}}\end{array}\right)=\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi\frac{\prod_{i=1}^r\phi_i u_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r\epsilon_{G_i}^{(i)}g_i!} \tag{1.1}$$

where

$$\phi_1=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-g_j+\sum_{i=1}^r\gamma_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\lambda'+1}^{A}\Gamma\left(g_j-\sum_{i=1}^r\gamma_j^{(i)}U_i\right)\prod_{j=1}^{C}\Gamma\left(1-f_j+\sum_{i=1}^r\xi_j^{(i)}\eta_{G_i,g_i}\right)} \tag{1.2}$$

$$\phi_i=\frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}}\Gamma\left(p_j^{(i)}-\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{\beta^{(i)}}\Gamma\left(1-q_j^{(i)}+\eta_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}}\Gamma\left(1-p_j^{(i)}+\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=\beta^{(i)}+1}^{M^{(i)}}\Gamma\left(q_j^{(i)}-\eta_j^{(i)}\eta_{G_i,g_i}\right)},i=1,\cdots,r \tag{1.3}$$

and

$$\eta_{G_i,g_i}=\frac{p_{G_i}^{(i)}+g_i}{\epsilon_{G_i}^{(i)}},i=1,\cdots,r \tag{1.4}$$

which is valid under the following conditions :

$$\epsilon_{G_i}^{(i)}[p_j^{(i)}+p_i']\neq\epsilon_j^{(i)}[p_{G_i}+g_i] \tag{1.5}$$

and

$$u_i \neq 0, \sum_{j=1}^{A} \gamma_j^{(i)} - \sum_{j=1}^{C} \xi_j^{(i)} + \sum_{j=1}^{M^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{N^{(i)}} \epsilon_j^{(i)} < 0, i = 1, \cdots, r \tag{1.6}$$

Here $\lambda, A, C, \alpha_i, \beta_i, m_i, n_i \in \mathbb{N}^*; i = 1, \cdots, r; f_j, g_j, p_j^{(i)}, q_j^{(i)}, \gamma_j^{(i)}, \xi_j^{(i)}, \eta_j^{(i)}, \epsilon_j^{(i)} \in \mathbb{C}$.

The multivariable A-function defined by Gautam and Asgar [3] is an extension of the multivariable H-function defined by Srivastava et al [4,5]. The multivariable A-function is defined in term of multiple Mellin-Barnes type integral.

$$A(Z_1, \cdots, Z_s) = A_{\mathbf{p},\mathbf{q}:p_1,q_1;\cdots;p_s,q_s}^{\mathbf{m},\mathbf{n}:m_1,n_1;\cdots;m_s,n_s} \left(\begin{array}{c|l} \mathrm{Z}_1 & (\mathrm{a}_j; A_j^{(1)}, \cdots, A_j^{(s)})_{1,\mathbf{p}} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{Z}_s & (\mathrm{b}_j; B_j^{(1)}, \cdots, B_j^{(s)})_{1,\mathbf{q}} : \end{array} \right.$$

$$\left. \begin{array}{l} (\mathrm{c}_j^{(1)}, C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(s)}, C_j^{(s)})_{1,p_s} \\ \\ (\mathrm{d}_j^{(1)}, D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(s)}, D_j^{(s)})_{1,q_s} \end{array} \right) \tag{1.7}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L_1'} \cdots \int_{L_s'} \phi'(u_1, \cdots, u_s) \prod_{i=1}^{s} \theta_i'(u_i) Z_i^{u_i} \mathrm{d}u_1 \cdots \mathrm{d}u_s \tag{1.8}$$

where $\phi'(u_1, \cdots, u_s), \theta_i'(u_i)$ for $i = 1, \cdots, s$ are given by :

$$\phi'(u_1, \cdots, u_s) = \frac{\prod_{j=1}^{\mathbf{m}} \Gamma(b_j - \sum_{i=1}^{s} B_j^{(i)} u_i) \prod_{j=1}^{\mathbf{n}} \Gamma(1 - a_j + \sum_{i=1}^{s} A_j^{(i)} u_j)}{\prod_{j=\mathbf{n}+1}^{\mathbf{p}} \Gamma(a_j - \sum_{i=1}^{s} A_j^{(i)} u_j) \prod_{j=\mathbf{m}+1}^{\mathbf{q}} \Gamma\left(1 - b_j + \sum_{i=1}^{s} B_j^{(i)} u_j\right)} \tag{1.9}$$

$$\theta_i'(u_i) = \frac{\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + C_j^{(i)} u_i) \prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - D_j^{(i)} u_i)}{\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - C_j^{(i)} u_i) \prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + D_j^{(i)} u_i)} \tag{1.10}$$

Here $\mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, m_i, n_i, p_i, c_i \in \mathbb{N}^*; i = 1, \cdots, r; a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i') Z_k| < \frac{1}{2} \eta_k' \pi, \xi'^* = 0, \eta_i' > 0 \tag{1.11}$$

$$\Omega_i' = \prod_{j=1}^{\mathbf{p}} \{A_j^{(i)}\}^{A_j^{(i)}} \prod_{j=1}^{\mathbf{q}} \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i} \{D_j^{(i)}\}^{D_j^{(i)}} \prod_{j=1}^{p_i} \{C_j^{(i)}\}^{-C_j^{(i)}}; i = 1, \cdots, s \tag{1.12}$$

$$\xi_i'^* = Im(\sum_{j=1}^{\mathbf{p}} A_j^{(i)} - \sum_{j=1}^{\mathbf{q}} B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)}); i = 1, \cdots, s \tag{1.13}$$

$$\eta_i' = Re\left(\sum_{j=1}^{\mathbf{n}} A_j^{(i)} - \sum_{j=\mathbf{n}+1}^{\mathbf{p}} A_j^{(i)} + \sum_{j=1}^{\mathbf{m}} B_j^{(i)} - \sum_{j=\mathbf{m}+1}^{\mathbf{q}} B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\right)$$

$$i = 1, \cdots, s \tag{1.14}$$

For convenience, we shall note.

$$X = m_1, n_1; \cdots ; m_s, n_s \quad : Y = p_1, q_1; \cdots ; p_s, q_s \tag{1.15}$$

$$\mathbb{A} = (a_j; A_j^{(1)}, \cdots, A_j^{(s)})_{1,\mathbf{p}} : \ (c_j^{(1)}, C_j^{(1)})_{1,p_1}; \cdots ; (c_j^{(s)}, C_j^{(s)})_{1,p_s} \tag{1.16}$$

$$\mathbb{B} = (b_j; B_j^{(1)}, \cdots, B_j^{(s)})_{1,\mathbf{q}} : \quad (d_j^{(1)}, D_j^{(1)})_{1,q_1}; \cdots ; (d_j^{(s)}, D_j^{(s)})_{1,q_s} \tag{1.17}$$

2. Multiple integral transformations

In this section, following multiple transformations are established.

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^p x_j^{s_j-1} \ A\begin{pmatrix} \mathrm{Z}_1 \left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_1} \left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_1} \prod_{j=1}^p x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ \mathrm{Z}_s \left(1+\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_s} \left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_s} \prod_{j=1}^p x_j^{v_j^{(s)}} \end{pmatrix}$$

$$f\left(\sum_{j=1}^p c_j'' x_j^{t_j}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_p = \prod_{j=1}^p \frac{(c_j'')^{-s_j/t_j}}{t_j} \int_0^\infty z^{S-1} f(z)$$

$$A_{\mathbf{p}+p,\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n}+p;X}\left(\begin{array}{c|c} \mathrm{Z}_1 \prod_{j=1}^p (c_j'')^{-v_j^{(1)}/t_j} z^{V_1+\mu_1}(1+z)^{v_1} & \left(1-\frac{s_j}{t_j} : \frac{v_j^{(1)}}{t_j}, \cdots, \frac{v_j^{(s)}}{t_j}\right)_{1,p}, \mathbb{A} \\ \cdot & \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s \prod_{j=1}^p (c_j'')^{-v_j^{(s)}/t_j} z^{V_s+\mu_s}(1+z)^{v_s} & (1\text{-}S: \mathrm{V}_1, \cdots, V_s), \mathbb{B} \end{array}\right) \mathrm{d}z \tag{2.1}$$

where $S = \sum_{j=1}^p \frac{s_j}{t_j}, V_i = \sum_{j=1}^p \frac{v_j^{(i)}}{t_j}; i = 1, \cdots, s$

provided that $\mu_i, v_j^{(i)} \geqslant 0; j = 1, \cdots, p, i = 1, \cdots, s ; v_i > 0; i = 1, \cdots, s$

$min\{c_j'', t_j, Re(s_j)\} > 0$ for $j = 1, \cdots, p$ and $|arg(\Omega_i')Z_k| < \frac{1}{2}\eta_k'\pi, \xi'^* = 0, \eta_i' > 0$

$v_i - \mu_i - V_i > 0, i = 1, \cdots, s$ and the function f is so prescribed that the various integrals (2.1) exist, also,

$$\int \cdots \int_{R_p} \prod_{j=1}^p x_j^{s_j-1} \ A\begin{pmatrix} \mathrm{Z}_1 \left(1-\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_1} \left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_1} \prod_{j=1}^p x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ \mathrm{Z}_s \left(1-\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{-v_s} \left(\sum_{j=1}^p c_j'' x_j^{t_j}\right)^{\mu_s} \prod_{j=1}^p x_j^{v_j^{(s)}} \end{pmatrix}$$

$$f\left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_p = \prod_{j=1}^{p} \frac{(c_j'')^{-s_j/t_j}}{t_j} \int_0^1 z^{S-1} f(z)$$

$$A^{\mathbf{m},\mathbf{n}+p;X}_{\mathbf{p}+p,\mathbf{q}+1;Y}\left(\begin{array}{c|c} Z_1 \prod_{j=1}^{p} (c_j'')^{-v_j^{(1)}/t_j} z^{V_1+\mu_1}(1-z)^{v_1} & \left(1-\frac{s_j}{t_j} : \frac{v_j^{(1)}}{t_j}, \cdots, \frac{v_j^{(s)}}{t_j}\right)_{1,p}, \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ Z_s \prod_{j=1}^{p} (c_j'')^{-v_j^{(s)}/t_j} z^{V_s+\mu_s}(1-z)^{v_s} & (\text{1-S}: V_1, \cdots, V_s), \mathbb{B} \end{array}\right) \mathrm{d}z \tag{2.2}$$

where $S = \sum_{j=1}^{p} \frac{s_j}{t_j}, V_i = \sum_{j=1}^{p} \frac{v_j^{(i)}}{t_j}; i = 1, \cdots, s$

provided that $v_i, \mu_i, v_j^{(i)} \geqslant 0; j = 1, \cdots, p, i = 1, \cdots, s$

$min\{c_j'', t_j, Re(s_j)\} > 0$ for $j = 1, \cdots, p$ and $|arg(\Omega_i')Z_k| < \frac{1}{2}\eta_k'\pi, \xi'^* = 0, \eta_i' > 0$

and the function f is so prescribed that the various integrals in (2.2) exist. Also R_p is the region defined by

$x_j \geqslant 0$ and $\sum_{i=1}^{p} c_j'' x^{t_j} \leqslant 1 (j = 1, \cdots, p)$

Proof of (2.1)

Writing contour integral for the multivariable A-function in the l.h.s. of (2.1), changing the order of integrations, which is justified under the conditions mentioned with (2.1), we find that

$$l.h.s. = \frac{1}{(2\pi\omega)^s} \int_{L_1'} \cdots \int_{L_s'} \phi'(u_1, \cdots, u_s) \prod_{i=1}^{s} \theta_i'(u_i) Z_i^{u_i} \left\{ \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{p} x_j^{s_j + \sum_{i=1}^{s} v_j^{(i)} u_i - 1}\right.$$

$$\left. \left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{\sum_{i=1}^{s} \mu_i u_i} \left(1 + \sum_{j=1}^{p} c_j'' x_j^{t_j}\right)^{-\sum_{i=1}^{s} v_i u_i} f\left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_p \right\} \mathrm{d}u_1 \cdots \mathrm{d}u_s \tag{2.3}$$

Now, by an appeal of the following useful result of Edwards [1, page 172] :

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{p} x_j^{s_j-1} f\left(\sum_{j=1}^{p} c_j'' x_j^{t_j}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_p = \frac{\prod_{j=1}^{p} \left(\frac{s_j}{t_j}\right)}{\Gamma\left(\sum_{j=1}^{p} \left(\frac{s_j}{t_j}\right)\right)} \prod_{j=1}^{p} \frac{(c_j'')^{-s_j/t_j}}{t_j} \int_0^\infty z^{S-1} f(z) \mathrm{d}z \tag{2.4}$$

where $S = \sum_{j=1}^{p} \frac{s_j}{t_j}; i = 1, \cdots, s$ and $min\{c_j'', t_j, Re(s_j)\} > 0$ for $j = 1, \cdots, p$

and using (1.8) again, we get the right hand side of (2.1).

To prove (2.2), we follow a method similar to that given above.

3. Multiple integrals

In the early section, we have remarked that a number of multiple integrals can be easily obtained by choosing the function f suitably. For the sake of illustration, we evaluate here following two simple and interesting multiple integrals :

First integral

Taking $f(z) = z^{\sigma}(1+z)^{-\lambda}$ in (2.1), using (1.8) and a known result [2, page 10], to evaluate the $z-$integral, we arrive at the following interesting and new integral.

$$\int_0^{\infty}\cdots\int_0^{\infty}\prod_{j=1}^{p}x_j^{s_j-1}\ A\begin{pmatrix} \mathrm{Z}_1\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^{p}x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ \mathrm{Z}_s\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^{p}x_j^{v_j^{(s)}} \end{pmatrix}$$

$$\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\sigma}\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-\lambda}\mathrm{d}x_1\cdots\mathrm{d}x_p=\prod_{j=1}^{p}\frac{(c_j'')^{-s_j/t_j}}{t_j}A^{\mathbf{m},\mathbf{n}+p+2;X}_{\mathbf{p}+p+2,\mathbf{q}+2;Y}\left(\begin{array}{c|} \mathrm{Z}_1\prod_{j=1}^{p}(c_j'')^{-v_j^{(1)}/t_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s\prod_{j=1}^{p}(c_j'')^{-v_j^{(s)}/t_j} \end{array}\right.$$

$$\left(1-\tfrac{s_j}{t_j}:\tfrac{v_j^{(1)}}{t_j},\cdots,\tfrac{v_j^{(s)}}{t_j}\right)_{1,p},(1-\sigma-S;\mu_1+V_1,\cdots,\mu_s+V_s),$$
$$\cdot$$
$$\cdot$$
$$(1\text{-}S\text{: } \mathrm{V}_1,\cdots,V_s),$$

$$\left.\begin{array}{c} (\ 1+\sigma-\lambda+S:v_1-\mu_1-V_1,\cdots,v_s-\mu_s-V_s),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\lambda;v_1,\cdots,v_s),\mathbb{B} \end{array}\right) \tag{3.1}$$

where $S=\sum_{j=1}^{p}\frac{s_j}{t_j},V_i=\sum_{j=1}^{p}\frac{v_j^{(i)}}{t_j};i=1,\cdots,s$

provided that $\mu_i,v_j^{(i)}\geqslant 0;j=1,\cdots,p,i=1,\cdots,s\,;v_i>0;i=1,\cdots,s$

$min\{c_j'',t_j,Re(s_j),Re(\sigma+1),Re(\lambda)\}>0$ for $j=1,\cdots,p$

$v_i-\mu_i-V_i>0,i=1,\cdots,s$ and $|arg(\Omega_i')Z_k|<\frac{1}{2}\eta_k'\pi,\xi'^{*}=0,\eta_i'>0$

$$0<Re(S+\sigma)+\sum_{i=1}^{s}(V_i+\mu_i)\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right)<Re(\lambda)+\sum_{i=1}^{s}v_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right)$$

Second integral

Taking $f(z)=z^{\sigma}(1-z)^{-\lambda}$ in (2.2), using (1.8) and a known result [2, page 10], to evaluate the $z-$integral, we arrive at the following interesting and new integral.

$$\int\cdots\int_{R_p}\prod_{j=1}^{p}x_j^{s_j-1}\quad A\left(\begin{array}{c}\mathrm{Z}_1\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^{p}x_j^{v_j^{(1)}}\\ \cdots\\ \cdots\\ \mathrm{Z}_s\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^{p}x_j^{v_j^{(s)}}\end{array}\right)$$

$$\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\sigma}\left(1-\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-\lambda}\mathrm{d}x_1\cdots\mathrm{d}x_p=\prod_{j=1}^{p}\frac{(c_j'')^{-s_j/t_j}}{t_j}$$

$$A^{\mathbf{m},\mathbf{n}+p+2;X}_{\mathbf{p}+p+2,\mathbf{q}+2;Y}\left(\begin{array}{c}\mathrm{Z}_1\prod_{j=1}^{p}(c_j'')^{-v_j^{(1)}/t_j}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{Z}_s\prod_{j=1}^{p}(c_j'')^{-v_j^{(s)}/t_j}\end{array}\left|\begin{array}{c}\left(1-\frac{s_j}{t_j}:\frac{v_j^{(1)}}{t_j},\cdots,\frac{v_j^{(s)}}{t_j}\right)_{1,p},\\ \cdot\\ \cdot\\ (1\text{-S}:\mathrm{V}_1,\cdots,V_s),\end{array}\right.\right.$$

$$\left.\begin{array}{c}(1\text{-}\sigma-S;\mu_1+V_1,\cdots,\mu_s+V_s),(-\lambda+S:v_1,\cdots,v_s),\mathbb{A}\\ \cdot\\ \cdot\\ (\text{-}\lambda-\sigma-S;\mu_1+v_1+V_1,\cdots,\mu_s+v_s+V_s),\mathbb{B}\end{array}\right) \tag{3.2}$$

where $S=\sum_{j=1}^{p}\frac{s_j}{t_j},V_i=\sum_{j=1}^{p}\frac{v_j^{(i)}}{t_j};i=1,\cdots,s$, also R_p is the region defined by

$x_j\geqslant 0$ and $\sum_{i=1}^{p}c_j''x^{t_j}\leqslant 1(j=1,\cdots,p)$

provided that $v_i,\mu_i,v_j^{(i)}\geqslant 0;j=1,\cdots,p,i=1,\cdots,s$

$min\{c_j'',t_j,Re(s_j),Re(\sigma+1),Re(\lambda+1)\}>0$ for $j=1,\cdots,p$

$v_i-\mu_i-V_i>0,i=1,\cdots,s$ and $|arg(\Omega_i')Z_k|<\frac{1}{2}\eta_k'\pi,\xi'^*=0,\eta_i'>0$

$0<Re(S+\sigma)+\sum_{i=1}^{s}(V_i+\mu_i)\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right)$ and $Re(\lambda+1)+\sum_{i=1}^{s}v_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right)>0$

4. Generalized of the integrals (3.1) and (3.2)

First integral

$$\int_0^{\infty}\cdots\int_0^{\infty}\prod_{j=1}^{p}x_j^{s_j-1}\quad\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\sigma}\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-\lambda}$$

$$A\left(\begin{array}{c}\mathrm{z}_1\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-u}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\rho}\prod_{j=1}^{p}x_j^{u_j^{(1)}}\\ \cdots\\ \cdots\\ \mathrm{z}_r\left(1+\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-u}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\rho}\prod_{j=1}^{p}x_j^{u_j^{(s)}}\end{array}\right)$$

$$A\left(\begin{array}{c} \mathrm{Z}_1\left(1+\sum_{j=1}^p c_j''x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^p c_j''x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^p x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ \mathrm{Z}_s\left(1+\sum_{j=1}^p c_j''x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^p c_j''x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^p x_j^{v_j^{(s)}} \end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_p$$

$$=\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi\frac{\prod_{i=1}^r\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r\epsilon_{G_i}^{(i)}g_i!}\prod_{j=1}^p\frac{(c_j'')^{-(s_j+u_j^{(i)}\eta_{G_i,g_i})/t_j}}{t_j}$$

$$A_{\mathbf{p}+p+2,\mathbf{q}+2;Y}^{\mathbf{m},\mathbf{n}+p+2;X}\left(\begin{array}{c|c} \mathrm{Z}_1\prod_{j=1}^p(c_j'')^{-v_j^{(1)}/t_j} & \left(1-\frac{s_j+u_j^{(i)}\eta_{G_i,g_i}}{t_j}:\frac{v_j^{(1)}}{t_j},\cdots,\frac{v_j^{(s)}}{t_j}\right)_{1,p}, \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_s\prod_{j=1}^p(c_j'')^{-v_j^{(s)}/t_j} & (1\text{-S}(\eta_{G_i,g_i}):V_1,\cdots,V_s), \end{array}\right.$$

$$(1\text{-}\sigma-\rho\eta_{G_i,g_i}-S(\eta_{G_i,g_i});\mu_1+V_1,\cdots,\mu_s+V_s),$$
$$\cdot$$
$$\cdot$$
$$\cdots$$

$$\left.\begin{array}{c} (\ 1\text{+}\sigma+\rho\eta_{G_i,g_i}-\lambda-\mu\eta_{G_i,g_i}+S(\eta_{G_i,g_i}):v_1-\mu_1-V_1,\cdots,v_s-\mu_s-V_s),\mathbb{A} \\ \cdot \\ \cdot \\ (1\text{-}\lambda-u\eta_{G_i,g_i}:v_1,\cdots,v_s),\mathbb{B} \end{array}\right) \quad (4.1)$$

where $S(\eta_{G_i,g_i})=\sum_{j=1}^p\frac{s_j+u_j^{(i)}\eta_{G_i,g_i}}{t_j}$, ϕ,ϕ_i and η_{G_i,g_i} are defined respectively by (1.2), (1.3) and (1.4).

The integral (4.1) is valid if :

(i) the sets of conditions given with (3.1) are satisfied.

(ii) $u=\rho-\sum_{j=1}^p\frac{u_j^{(i)}}{t_j}>0,u,\rho,u_j^{(i)}\geqslant 0(j=1,\cdots,p;i=1,\cdots,r)$

(iii) The multiple series on the right hand side are absolutely convergents.

(iv) $u_i\neq 0,\sum_{j=1}^A\gamma_j^{(i)}-\sum_{j=1}^C\xi_j^{(i)}+\sum_{j=1}^{M^{(i)}}\eta_j^{(i)}-\sum_{j=1}^{N^{(i)}}\epsilon_j^{(i)}<0,i=1,\cdots,r$

Second integral

$$\int\cdots\int_{R_p}\prod_{j=1}^p x_j^{s_j-1}\left(\sum_{j=1}^p c_j''x_j^{t_j}\right)^{\sigma}\left(1-\sum_{j=1}^p c_j''x_j^{t_j}\right)^{-\lambda}$$

$$A\left(\begin{array}{c} \mathrm{z}_1\left(1-\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-u}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\rho}\prod_{j=1}^{p}x_j^{u_j^{(1)}} \\ \cdots \\ \cdots \\ \mathrm{z}_r\left(1-\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-u}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\rho}\prod_{j=1}^{p}x_j^{u_j^{(s)}} \end{array}\right)$$

$$A\left(\begin{array}{c} \mathrm{Z}_1\left(1-\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-v_1}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\mu_1}\prod_{j=1}^{p}x_j^{v_j^{(1)}} \\ \cdots \\ \cdots \\ \mathrm{Z}_s\left(1-\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{-v_s}\left(\sum_{j=1}^{p}c_j''x_j^{t_j}\right)^{\mu_s}\prod_{j=1}^{p}x_j^{v_j^{(s)}} \end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_p$$

$$=\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi\frac{\prod_{i=1}^{r}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{r}g_i}}{\prod_{i=1}^{r}\epsilon_{G_i}^{(i)}g_i!}\prod_{j=1}^{p}\frac{(c_j'')^{-(s_j+u_j^{(i)}\eta_{G_i,g_i})/t_j}}{t_j}$$

$$A_{\mathbf{p}+p+2,\mathbf{q}+2;Y}^{\mathbf{m},\mathbf{n}+p+2;X}\left(\begin{array}{c} \mathrm{Z}_1\prod_{j=1}^{p}(c_j'')^{-v_j^{(1)}/t_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s\prod_{j=1}^{p}(c_j'')^{-v_j^{(s)}/t_j} \end{array}\middle| \begin{array}{c} \left(1-\frac{s_j+u_j^{(i)}\eta_{G_i,g_i}}{t_j}:\frac{v_j^{(1)}}{t_j},\cdots,\frac{v_j^{(s)}}{t_j}\right)_{1,p}, \\ \cdot \\ \cdot \\ (1\text{-S}(\eta_{G_i,g_i}):V_1,\cdots,V_s), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\sigma-\rho\eta_{G_i,g_i}-S(\eta_{G_i,g_i});\mu_1+V_1,\cdots,\mu_s+V_s),(-\lambda-u\eta_{G_i,g_i}:v_1,\cdots,v_s),\mathbb{A} \\ \cdot \\ \cdot \\ (\text{-}\lambda-\sigma-(u+\rho)\eta_{G_i,g_i}-S(\eta_{G_i,g_i}):\mu_1+v_1+V_1,\cdots,\mu_s+v_s+V_s),\mathbb{B} \end{array}\right) \tag{4.2}$$

where $S(\eta_{G_i,g_i})=\sum_{j=1}^{p}\frac{s_j+u_j^{(i)}\eta_{G_i,g_i}}{t_j}$, ϕ,ϕ_i and η_{G_i,g_i} are defined respectively by (1.2), (1.3) and (1.4).

also R_p is the region defined by $x_j\geqslant 0$ and $\sum_{i=1}^{p}c_j''x^{t_j}\leqslant 1(j=1,\cdots,p)$

The integral (4.2) is valid if :

(i) the sets of conditions given with (3.2) are satisfied.

(ii) $u,\rho,u_j^{(i)}\geqslant 0(j=1,\cdots,p;i=1,\cdots,r)$

(iii) The multiple series on the right hand side are absolutely convergents.

(iv) $u_i\neq 0,\sum_{j=1}^{A}\gamma_j^{(i)}-\sum_{j=1}^{C}\xi_j^{(i)}+\sum_{j=1}^{M^{(i)}}\eta_j^{(i)}-\sum_{j=1}^{N^{(i)}}\epsilon_j^{(i)}<0,i=1,\cdots,r$

Proof

To prove (4.1), first expressing the multivariable A-function defined by Gautam and Asgar [] in series with the help of (1.1) and we interchange the order of summations and $(x_1\cdots x_p)-$integral (which is permissible under the conditions stated). Now collect the power of $\sum_{j=1}^{p} c_j'' x_j^{t_j}$ and $1 - \sum_{j=1}^{p} c_j'' x_j^{t_j}$ and use the integral (3.1), we obtain the desired result.

The integral (4.2) can be proved in a similar manner.

5. Conclusion

The integral (4.1) and (4.2) involve the multivariable A-functions defined by Gautam and Asgar [3], which is quite general in character. On specializing the various parameters of these functions, a number of new obtained double, triple and multiple integrals can be obtained involving special functions of one and several variables.

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