On Total Regularity of the Cartesian product of two Interval – valued Fuzzy Graphs

Ann Mary Philp
Department of Mathematics, Assumption College, Changanassery – 686101, Kerala, India.

Abstract: *In this paper, we define the total degree of a vertex in the Cartesian product of two interval – valued fuzzy graphs (IVFG) and investigate the total regularity of the Cartesian product. In general, the Cartesian product of two totally regular interval – valued fuzzy graphs need not be a totally regular interval – valued fuzzy graph (TRIVFG). The necessary and sufficient conditions for the Cartesian product of two TRIVFGs to be totally regular under some restrictions are obtained.*

Keywords: *Interval – valued fuzzy graph, Cartesian Product, Total regularity.*

I. Introduction

Graph theory has so many applications in almost all real world problems. But since the world is full of uncertainty, fuzzy graph has a separate importance in many real life applications. Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. In the applied field, the success of the use of fuzzy set theory depends on the choice of the membership function that we make. However, there are applications in which experts do not have precise knowledge of the function that should be taken. In these cases, it is appropriate to represent the membership degree of each element of the fuzzy set by means of an interval. From these considerations arises the extension of fuzzy sets called the theory of interval- valued fuzzy sets, that is fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval [0,1]. Replacing the membership functions of vertices and edges in fuzzy graphs by interval valued fuzzy sets such that they satisfy some particular condition, interval – valued fuzzy graphs were defined. Thus IVFG provide a more description of vagueness and uncertainty within the specific interval than the traditional fuzzy graph. The basic concepts of fuzzy sets and interval – valued fuzzy sets can be found in [43] and [44].

The first definition of fuzzy graph was by Kaufmann [14] in 1973. But it was Azriel Rosenfeld [34] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs as a generalization of Eulers graph theory in 1975. The works of Bhattacharya[7], Bhutani [8], Bhutani and Battou [9], Bhutani and Rosenfeld [10]-[12], Mordeson [15], Mordeson and Nair [16],[17], Mordeson and Peng [18], Sunitha and Vijayakumar[37]-[40], Nagoor Gani and Basheer Ahmed [19], Nagoor Gani and Malarvizhi[20], Nagoor Gani and Radha[21],[22] form the foundation of all researches in fuzzy graph theory.

In 2009, Hongmei and Lianhua [13] introduced IVFG as an extension of fuzzy graphs. Since then, a lot of research work is being done in this area. Muhammad Akram and Wieslaw A. Dudek [3] defined the operations of Cartesian product, composition, union and join of IVFGS and investigated some properties. They also introduced the notion of interval valued fuzzy complete graphs and presented some properties of self complementary and self weak complementary interval valued fuzzy complete graphs. Now IVFG is growing fast and has wide applications in many fields. The various works done in this area can be seen in [1] – [6],[25] – [33], [35] – [36] and [41] – [42]. H. Rashmanlou and Madhumangal Pal [25] defined regular and totally regular IVFGs. Total regularity of the join of two IVFGs was discussed by the author in [35]. The author also studied about regular and edge regular IVFGs [36]. Totally regular property of the Cartesian Product of two intuitionistic fuzzy graphs were studied by A. Nagoor Gani and H. Sheik Mujibur Rahman[23],[24].

In this paper, we introduce and analyse the notion of total degree of a vertex in the Cartesian Product of two IVFGs. Also we obtain the necessary and sufficient conditions for the Cartesian product of two TRIVFG to be totally regular under some restrictions.

II. Basic Concepts

Graph theoretic terms and results used in this work are either standard or are explained as and when they first appear. We consider only simple graphs. That is, graphs with multiple edges and loops are not considered.

Definition 2.1[34].

Let V be a non empty set. A fuzzy graph is a pair of functions $G:(\sigma,\mu)$ where σ is a fuzzy subset of V and μ is a symmetric fuzzy relation on σ. That is, $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ such that $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$ for all u,v in V where $\sigma(u) \wedge \sigma(v)$ denotes minimum of $\sigma(u)$ *and* $\sigma(v)$.

Definition 2.2[3].

An interval number D is an interval $[a^-, a^+]$ with $0 \le a^- \le a^+ \le 1$.

Remark 2.1.

(i) The interval number $[a, a]$ is identified with the number $a \in [0,1]$.

(ii) $D[0,1]$ denotes the set of all interval numbers.

Definition 2.3[3].

For interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$

- $\mathrm{rmin}(D_1, D_2) = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$
- $\mathrm{rmax}(D_1, D_2) = [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$
- $D_1 + D_2 = [a_1^- + a_2^- - a_1^- . a_2^-,\ b_1^+ + b_2^+ - b_1^+ . b_2^+]$
- $D_1 \le D_2 \Leftrightarrow a_1^- \le a_2^-$ and $b_1^+ \le b_2^+$
- $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$
- $D_1 < D_2 \Leftrightarrow D_1 \le D_2$ and $D_1 \ne D_2$
- $kD = k[a_1^-, b_1^+] = [ka_1^-, kb_1^+]$ where $0 \le k \le 1$.

Then $(D[0,1], \le, \vee, \wedge)$ is a complete lattice with $[0,0]$ as the least element and $[1,1]$ as the greatest. Here $\vee$ denotes *maximum* and $\wedge$ denotes *minimum*.

Definition 2.4[3].

The **i**nterval – valued fuzzy set (IVFS) A in V is defined by $A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in V\}$ where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of V such that $\mu_A^-(x) \le \mu_A^+(x)$ for all $x \in V$. We shall sometimes denote the IVFS A by $[\mu_A^-(x), \mu_A^+(x)]$.

For any two IVFSs $A = [\mu_A^-(x), \mu_A^+(x)]$ and $B = [\mu_B^-(x), \mu_B^+(x)]$ in V, we define

- $A \cup B = \left\{\left(x, \max\left(\mu_A^-(x), \mu_B^-(x)\right), \max\left(\mu_A^+(x), \mu_B^+(x)\right)\right) : x \in V\right\}$
- $A \cap B = \left\{\left(x, min\left(\mu_A^-(x), \mu_B^-(x)\right), min\left(\mu_A^+(x), \mu_B^+(x)\right)\right) : x \in V\right\}$

Definition 2.5[3].

If $G^* = (V, E)$ is a graph, then by an interval – valued fuzzy relation (IVFR) B on the set E we mean an IVFS such that $\mu_B^-(xy) \le min\left(\mu_A^-(x), \mu_A^-(y)\right)$ and $\mu_B^+(xy) \le min\left(\mu_A^+(x), \mu_A^+(y)\right)$ for all $xy \in E$.

Definition 2.6 [3].

By an interval – valued fuzzy graph (IVFG) of a graph $G^* = (V, E)$, we mean a pair $G = (A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an IVFS on V and $B = [\mu_B^-, \mu_B^+]$ is an IVFR on E.

***Definition 2.7** [25]*.

The negative degree of a vertex $u \in V$ is defined by $d^-(u) = \sum_{uv \in E} \mu_B^-(uv)$. Similarly, positive degree of a vertex $u \in V$ is defined by $d^+(u) = \sum_{uv \in E} \mu_B^+(uv)$. Then the degree of the vertex $u \in V$ is defined as $d(u) = [d^-(u), d^+(u)]$.

***Definition 2.8** [25]*.

If $d^-(u) = k_1, d^+(u) = k_2$ for all $u \in V$ where k_1, k_2 are real numbers, then the graph G is called $[k_1, k_2]$ - regular interval – valued fuzzy graph(RIVFG) or regular interval – valued fuzzy graph of degree $[k_1, k_2]$.

Definition 2.9[25].

The total degree of the vertex $u \in V$ is defined as $td(u) = [td^-(u), td^+(u)]$ where,

$$td^-(u) = \sum_{uv \in E} \mu_B^-(uv) + \mu_A^-(u) = d^-(u) + \mu_A^-(u)$$ and

$$td^+(u) = \sum_{uv \in E} \mu_B^+(uv) + \mu_A^+(u) = d^+(u) + \mu_A^+(u).$$

Definition 2.10[25].

Let $G = (A, B)$ be an IVFG. If each vertex of G has the same total degree $[k_1, k_2]$ then the graph G is called a $[k_1, k_2]$ totally regular interval valued fuzzy graph (TRIVFG) or TRIVFG of degree $[k_1, k_2]$.

Definition 2.11[36].

If the underlying graph G^* is regular, then G is said to be a partially regular interval – valued fuzzy graph (PRIVFG).

Definition 2.12[3].

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs with $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then the Cartesian product $G_1 \times G_2$ of G_1 and G_2 is a pair of functions $(A_1 \times A_2, B_1 \times B_2)$ with underlying vertex set $V_1 \times V_2 = \{(u_1, v_1) : u_1 \in V_1\ and\ v_1 \in V_2\}$ and underlying edge set $E_1 \times E_2 = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2, v_1v_2 \in E_2\ or\ u_1u_2 \in E_1, v_1 = v_2\}$ such that

$$i. \begin{cases} (\mu_{A_1}^- \times \mu_{A_2}^-)(u_1, v_1) = \min\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right) \\ (\mu_{A_1}^+ \times \mu_{A_2}^+)(u_1, v_1) = \min\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right) \end{cases}$$

for all $(u_1, v_1) \in V = V_1 \times V_2$.

$$ii \begin{cases} (\mu_{B_1}^- \times \mu_{B_2}^-)((u_1, v_1)(u_2, v_2)) \\ = \min\left(\mu_{A_1}^-(u_1), \mu_{B_2}^-(v_1v_2)\right) \\ (\mu_{B_1}^+ \times \mu_{B_2}^+)((u_1, v_1)(u_2, v_2)) \\ = \min\left(\mu_{A_1}^+(u_1), \mu_{B_2}^+(v_1v_2)\right) \end{cases}$$

if $u_1 = u_2, v_1v_2 \in E_2$

$$iii \begin{cases} (\mu_{B_1}^- \times \mu_{B_2}^-)((u_1,v_1)(u_2,v_2)) \\ = \min\left(\mu_{B_1}^-(u_1u_2), \mu_{A_2}^-(v_1)\right) \\ (\mu_{B_1}^+ \times \mu_{B_2}^+)((u_1,v_1)(u_2,v_2)) \\ = \min\left(\mu_{B_1}^+(u_1u_2), \mu_{A_2}^+(v_1)\right) \end{cases}$$

if $u_1u_2 \in E_1, v_1 = v_2$

Remark 2.1.

Clearly $G_1 \times G_2$ is an IVFG.

III. Total Degree of a Vertex in Cartesian Product

By definition, for any $(u_1,v_1) \in V_1 \times V_2$,

$td_{G_1 \times G_2}^-(u_1,v_1)$

$= \sum_{(u_1,v_1)(u_2,v_2)\in E_1\times E_2}(\mu_{B_1}^- \times \mu_{B_2}^-)((u_1,v_1)(u_2,v_2)) + (\mu_{A_1}^- \times \mu_{A_2}^-)(u_1,v_1)$

$= \sum_{u_1=u_2, v_1v_2\in E_2} \min\left(\mu_{A_1}^-(u_1), \mu_{B_2}^-(v_1v_2)\right) + \sum_{u_1u_2\in E_1, v_1=v_2} \min\left(\mu_{B_1}^-(u_1u_2), \mu_{A_2}^-(v_1)\right) + \min\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$.............. **(3.1)**

Also,

$td_{G_1 \times G_2}^-(u_1,v_1)$

$= \sum_{(u_1,v_1)(u_2,v_2)\in E_1\times E_2}(\mu_{B_1}^- \times \mu_{B_2}^-)((u_1,v_1)(u_2,v_2)) + (\mu_{A_1}^- \times \mu_{A_2}^-)(u_1,v_1)$

$= \sum_{u_1=u_2, v_1v_2\in E_2} \min\left(\mu_{A_1}^+(u_1), \mu_{B_2}^+(v_1v_2)\right) + \sum_{u_1u_2\in E_1, v_1=v_2} \min\left(\mu_{B_1}^+(u_1u_2), \mu_{A_2}^+(v_1)\right) + \min\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right)$.............. **(3.2)**

Result 3.1

For any two numbers a and b,

$\min(a,b) = a + b - \max(a,b)$

Using the above result, equations (3.1) and (3.2) can be rewritten as

$td_{G_1 \times G_2}^-(u_1,v_1)$

$= \sum_{u_1=u_2, v_1v_2\in E_2} \min\left(\mu_{A_1}^-(u_1), \mu_{B_2}^-(v_1v_2)\right) + \sum_{u_1u_2\in E_1, v_1=v_2} \min\left(\mu_{B_1}^-(u_1u_2), \mu_{A_2}^-(v_1)\right) + \mu_{A_1}^-(u_1) + \mu_{A_2}^-(v_1) - \max\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$ **(3.3)**

Also,

$td_{G_1 \times G_2}^+(u_1,v_1)$

$= \sum_{u_1=u_2, v_1v_2\in E_2} \min\left(\mu_{A_1}^+(u_1), \mu_{B_2}^+(v_1v_2)\right) + \sum_{u_1u_2\in E_1, v_1=v_2} \min\left(\mu_{B_1}^+(u_1u_2), \mu_{A_2}^+(v_1)\right) + \mu_{A_1}^+(u_1) + \mu_{A_2}^+(v_1) - \max\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right)$ **(3.4)**

Now we find the total degree of a vertex (u_1,v_1) in $G_1 \times G_2$ in some particular cases.

Theorem 3.1.

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \geq B_2$ and $A_2 \geq B_1$, then

$td_{G_1 \times G_2}^-(u_1,v_1) = d_{G_1}^-(u_1) + d_{G_2}^-(v_1) + min\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$ and

$td_{G_1 \times G_2}^+(u_1,v_1) = d_{G_1}^+(u_1) + d_{G_2}^+(v_1) + min\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right)$

Proof:

$A_1 \geq B_2 \Rightarrow [\mu_{A_1}^-, \mu_{A_1}^+] \geq [\mu_{B_2}^-, \mu_{B_2}^+]$

$\Rightarrow \mu_{A_1}^- \geq \mu_{B_2}^-$ *and* $\mu_{A_1}^+ \geq \mu_{B_2}^+$

Again, $A_2 \geq B_1 \Rightarrow [\mu_{A_2}^-, \mu_{A_2}^+] \geq [\mu_{B_1}^-, \mu_{B_1}^+]$

$\Rightarrow \mu_{A_2}^- \geq \mu_{B_1}^-$ *and* $\mu_{A_2}^+ \geq \mu_{B_1}^+$

$\therefore \min\left(\mu_{A_1}^-(u_1), \mu_{B_2}^-(v_1v_2)\right) = \mu_{B_2}^-(v_1v_2)$ and

$\min\left(\mu_{B_1}^-(u_1u_2), \mu_{A_2}^-(v_1)\right) = \mu_{B_1}^-(u_1u_2)$

Similarly, $\min\left(\mu_{A_1}^+(u_1), \mu_{B_2}^+(v_1v_2)\right) = \mu_{B_2}^+(v_1v_2)$

and $\min\left(\mu_{B_1}^+(u_1u_2), \mu_{A_2}^+(v_1)\right) = \mu_{B_1}^+(u_1u_2)$

Then from equation (3.1), we have

$\therefore td_{G_1 \times G_2}^-(u_1,v_1) = \sum_{v_1v_2\in E_2} \mu_{B_2}^-(v_1v_2) + \sum_{u_1u_2\in E_1} \mu_{B_1}^-(u_1u_2) + \min\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$

$= d_{G_1}^-(u_1) + d_{G_2}^-(v_1) + min\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$

Again from equation (3.2), we have

$td_{G_1 \times G_2}^+(u_1,v_1) = \sum_{v_1v_2\in E_2} \mu_{B_2}^+(v_1v_2) + \sum_{u_1u_2\in E_1} \mu_{B_1}^+(u_1u_2) + min\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right)$

$= d_{G_1}^+(u_1) + d_{G_2}^+(v_1) + min\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right)$.

Theorem 3.2.

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \geq B_2$ and $A_2 \geq B_1$, then

$td_{G_1 \times G_2}^-(u_1,v_1) = td_{G_1}^-(u_1) + td_{G_2}^-(v_1) - max\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$ and

$td_{G_1 \times G_2}^+(u_1,v_1) = td_{G_1}^+(u_1) + td_{G_2}^+(v_1) - max\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right)$

Proof:

$A_1 \geq B_2 \Rightarrow [\mu_{A_1}^-, \mu_{A_1}^+] \geq [\mu_{B_2}^-, \mu_{B_2}^+]$

$\Rightarrow \mu_{A_1}^- \geq \mu_{B_2}^-$ *and* $\mu_{A_1}^+ \geq \mu_{B_2}^+$

Again, $A_2 \geq B_1 \Rightarrow [\mu_{A_2}^-, \mu_{A_2}^+] \geq [\mu_{B_1}^-, \mu_{B_1}^+]$

$\Rightarrow \mu_{A_2}^- \geq \mu_{B_1}^-$ *and* $\mu_{A_2}^+ \geq \mu_{B_1}^+$

$\therefore \min\left(\mu_{A_1}^-(u_1), \mu_{B_2}^-(v_1v_2)\right) = \mu_{B_2}^-(v_1v_2)$ and

$\min\left(\mu_{B_1}^-(u_1u_2), \mu_{A_2}^-(v_1)\right) = \mu_{B_1}^-(u_1u_2)$

Similarly, $\min\left(\mu_{A_1}^+(u_1), \mu_{B_2}^+(v_1v_2)\right) = \mu_{B_2}^+(v_1v_2)$

and $\min\left(\mu_{B_1}^+(u_1u_2), \mu_{A_2}^+(v_1)\right) = \mu_{B_1}^+(u_1u_2)$

Then from equation (3.3), we have

$td_{G_1 \times G_2}(u_1,v_1)$

$= \sum_{v_1v_2\in E_2} \mu_{B_2}^-(v_1v_2) + \sum_{u_1u_2\in E_1} \mu_{B_1}^-(u_1u_2) + \mu_{A_1}^-(u_1) + \mu_{A_2}^-(v_1) - \max\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$

$= \sum_{u_1u_2\in E_1} \mu_{B_1}^-(u_1u_2) + \mu_{A_1}^-(u_1) + \sum_{v_1v_2\in E_2} \mu_{B_2}^-(v_1v_2) + \mu_{A_2}^-(v_1) - \max\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$

$= td^-_{G_1}(u_1) + td^-_{G_2}(v_1) - max\left(\mu^-_{A_1}(u_1), \mu^-_{A_2}(v_1)\right)$
Again from equation (3.4), we have
$td^+_{G_1\times G_2}(u_1, v_1)$
$= \sum_{v_1v_2\in E_2} \mu^+_{B_2}(v_1v_2) + \sum_{u_1u_2\in E_1} \mu^+_{B_1}(u_1u_2) + \mu^+_{A_1}(u_1) + \mu^+_{A_2}(v_1) - max\left(\mu^+_{A_1}(u_1), \mu^+_{A_2}(v_1)\right)$
$= \sum_{u_1u_2\in E_1} \mu^+_{B_1}(u_1u_2) + \mu^+_{A_1}(u_1) + \sum_{v_1v_2\in E_2} \mu^+_{B_2}(v_1v_2) + \mu^+_{A_2}(v_1) - \max\left(\mu^+_{A_1}(u_1), \mu^+_{A_2}(v_1)\right)$
$= td^+_{G_1}(u_1) + td^+_{G_2}(v_1) - max\left(\mu^+_{A_1}(u_1), \mu^+_{A_2}(v_1)\right)$

Lemma 3.1.
Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \le B_2$, then $A_2 \ge B_1$ and vice versa.

Proof:
By the definition of an IVFG,
$\mu^-_{B_i}(u, v) \le min\left(\mu^-_{A_i}(u), \mu^-_{A_i}(v)\right)$ *and*
$\mu^+_{B_i}(u, v) \le min\left(\mu^+_{A_i}(u), \mu^+_{A_i}(v)\right)$ for all $(u, v) \in E_i$ where i = 1,2.
$\therefore\ \mu^-_{B_i} \le max\mu^-_{A_i}$ and $min\mu^-_{B_i} \le \mu^-_{A_i}$ …..**(3.5)** for i=1,2
Now, $A_1 \le B_2 \Rightarrow [\mu^-_{A_1}, \mu^+_{A_1}] \le [\mu^-_{B_2}, \mu^+_{B_2}]$
$\Rightarrow \mu^-_{A_1} \le \mu^-_{B_2}$ *and* $\mu^+_{A_1} \le \mu^+_{B_2}$
Again, $\mu^-_{A_1} \le \mu^-_{B_2} \Rightarrow \max\mu^-_{A_1} \le \min\mu^-_{B_2}$ … **(3.6)**
Then from (3.5) and (3.6), we have
$\mu^-_{B_1} \le max\mu^-_{A_1} \le \min\mu^-_{B_2} \le \mu^-_{A_2}$
Similarly, $\mu^+_{B_1} \le \mu^+_{A_2}$
$\therefore\ [\mu^-_{B_1}, \mu^+_{B_1}] \le [\mu^-_{A_2}, \mu^+_{A_2}]$
i.e. $B_1 \le A_2$ or in otherwords $A_2 \ge B_1$

Lemma 3.2.
Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $A_1 \le B_2$. Then $A_1 \le A_2$.

Proof:
By the definition an IVFG,
$\mu^-_{B_i}(u, v) \le min\left(\mu^-_{A_i}(u), \mu^-_{A_i}(v)\right)$ *and*
$\mu^+_{B_i}(u, v) \le min\left(\mu^+_{A_i}(u), \mu^+_{A_i}(v)\right)$ for all $(u, v) \in E_i$ where i = 1,2. In particular when i=2,
$\mu^-_{B_2}(u, v) \le min\left(\mu^-_{A_2}(u), \mu^-_{A_2}(v)\right)$ and
$\mu^+_{B_2}(u, v) \le min\left(\mu^+_{A_2}(u), \mu^+_{A_2}(v)\right)$ for all $(u, v) \in E_2$
$\therefore \min\mu^-_{B_2} \le \mu^-_{A_2}$ and $\min\mu^+_{B_2} \le \mu^+_{A_2}$
Now, $A_1 \le B_2 \Rightarrow [\mu^-_{A_1}, \mu^+_{A_1}] \le [\mu^-_{B_2}, \mu^+_{B_2}]$
$\Rightarrow \mu^-_{A_1} \le \mu^-_{B_2}$ *and* $\mu^+_{A_1} \le \mu^+_{B_2}$
Then, $\mu^-_{A_1} \le \mu^-_{B_2} \Rightarrow \mu^-_{A_1} \le \min\mu^-_{B_2} \le \mu^-_{A_2}$
$\therefore \mu^-_{A_1} \le \mu^-_{A_2}$
Similarly, $\mu^+_{A_1} \le \mu^+_{A_2}$
$\therefore\ [\mu^-_{A_1}, \mu^+_{A_1}] \le [\mu^-_{A_2}, \mu^+_{A_2}]$
i.e. $A_1 \le A_2$

Theorem 3.3.
Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $A_1 \le B_2$. Then, $td^-_{G_1\times G_2}(u_1, v_1) = d^-_{G_1}(u_1) + \mu^-_{A_1}(u_1)dG^*_2(v_1) + \mu^-_{A_1}(u_1)$ and $td^+_{G_1\times G_2}(u_1, v_1) = d^+_{G_1}(u_1) + \mu^+_{A_1}(u_1)dG^*_2(v_1) + \mu^+_{A_1}(u_1)$

Proof:
Since $A_1 \le B_2$, by lemmas 3.1 and 3.2, we have $A_2 \ge B_1$ and $A_1 \le A_2$ respectively. Using these conditions in equation 3.1, $td^-_{G_1\times G_2}(u_1, v_1)$
$= \sum_{v_1v_2\in E_2} \mu^-_{A_1}(u_1) + \sum_{u_1u_2\in E_1} \mu^-_{B_1}(u_1u_2) + \mu^-_{A_1}(u_1)$
$= \sum_{u_1u_2\in E_1} \mu^-_{B_1}(u_1u_2) + \mu^-_{A_1}(u_1)\sum_{v_1v_2\in E_2} 1 + \mu^-_{A_1}(u_1)$
$= d^-_{G_1}(u_1) + \mu^-_{A_1}(u_1)dG^*_2(v_1) + \mu^-_{A_1}(u_1)$
Similarly, $td^+_{G_1\times G_2}(u_1, v_1) = d^+_{G_1}(u_1) + \mu^+_{A_1}(u_1)dG^*_2(v_1) + \mu^+_{A_1}(u_1)$

Theorem 3.4.
Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $A_1 \le B_2$. Then, $td^-_{G_1\times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + \mu^-_{A_1}(u_1)dG^*_2(v_1)$ and
$td^+_{G_1\times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + \mu^+_{A_1}(u_1)dG^*_2(v_1)$

Proof:
Since $A_1 \le B_2$, by lemmas 3.1 and 3.2, we have $A_2 \ge B_1$ and $A_1 \le A_2$ respectively. Using these conditions in equation 3.1, $td^-_{G_1\times G_2}(u_1, v_1)$
$= \sum_{v_1v_2\in E_2} \mu^-_{A_1}(u_1) + \sum_{u_1u_2\in E_1} \mu^-_{B_1}(u_1u_2) + \mu^-_{A_1}(u_1)$
$= \sum_{u_1u_2\in E_1} \mu^-_{B_1}(u_1u_2) + \mu^-_{A_1}(u_1)\sum_{v_1v_2\in E_2} 1 + \mu^-_{A_1}(u_1)$
$= d^-_{G_1}(u_1) + \mu^-_{A_1}(u_1) + \mu^-_{A_1}(u_1)dG^*_2(v_1)$
$= td^-_{G_1}(u_1) + \mu^-_{A_1}(u_1)dG^*_2(v_1)$
Similarly,
$td^+_{G_1\times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + \mu^+_{A_1}(u_1)dG^*_2(v_1)$.

IV. TOTAL REGULARITY OF THE CARTESIAN PRODUCT OF TWO INTERVAL-VALUED FUZZY GRAPHS

In general the Cartesian product of two TRIVFGs need not be a TRIVFG which is clear from the following example.

Example 4.1.
Consider the IVFG $G_1 = (A_1, B_1)$ defined on graph $G^*_1 = (V_1, E_1)$ such that $V_1 = \{u_1, u_2, u_3\}$ and $E_1 = \{u_1u_2, u_2u_3, u_1u_3\}$. The membership degrees of the vertices and edges are as follows:

	u_1	u_2	u_3
$\mu_{A_1}^-$	0.3	0.5	0.5
$\mu_{A_1}^+$	0.5	0.6	0.6

	u_1u_2	u_2u_3	u_1u_3
$\mu_{B_1}^-$	0.3	0.1	0.3
$\mu_{B_1}^+$	0.5	0.4	0.5

Now consider the IVFG $G_2 = (A_2, B_2)$ defined by

	v_1	v_2
$\mu_{A_2}^-$	0.4	0.4
$\mu_{A_1}^+$	0.5	0.5

	v_1v_2
$\mu_{B_2}^-$	0.4
$\mu_{B_2}^+$	0.5

Then $G_1 \times G_2$ will be an IVFG defined by

	$(u_1, v_1$	$(u_1, v_2$	$(u_2, v_1$	$(u_2, v_2$	$(u_3, v_1$	$(u_3, v_2$
$\mu_{A_1}^- \times \mu_{A_2}^-$	0.3	0.3	0.4	0.4	0.4	0.4
$\mu_{A_1}^+ \times \mu_{A_2}^+$	0.5	0.5	0.5	0.5	0.5	0.5

	$(u_1, v_1)(u_1, v_2$	$(u_1, v_1)(u_2, v_1$	$(u_1, v_1)(u_3, v_1$
$\mu_{B_1}^- \times \mu_{B_2}^-$	0.3	0.3	0.3
$\mu_{B_1}^+ \times \mu_{B_2}^+$	0.5	0.5	0.5
	$(u_1, v_2)(u_2, v_2$	$(u_1, v_2)(u_3, v_2$	$(u_2, v_1)(u_2, v_2$
	0.3	0.3	0.4
	0.5	0.5	0.5
	$(u_2, v_1)(u_3, v_1$	$(u_2, v_2)(u_3, v_2$	$(u_3, v_1)(u_3, v_2$
	0.1	0.1	0.4
	0.4	0.4	0.5

Using routine computations, we can see that both G_1 *and* G_2 are TRIVFGs, but $G_1 \times G_2$ is not a TRIVFG.

Next we give an example to show that $G_1 \times G_2$ is a TRIVFG need not imply that both G_1 *and* G_2 should be TRIVFGs

Example 4.2.

Consider the IVFG $G_2 = (A_1, B_1)$ defined by

	u_1	u_2
$\mu_{A_1}^-$	0.4	0.4
$\mu_{A_1}^+$	0.5	0.5

	u_1u_2
$\mu_{B_1}^-$	0.3
$\mu_{B_1}^+$	0.4

Also consider the IVFG $G_2 = (A_2, B_2)$ defined by

	v_1	v_2
$\mu_{A_2}^-$	0.5	0.4
$\mu_{A_1}^+$	0.6	0.5

	v_1v_2
$\mu_{B_2}^-$	0.2
$\mu_{B_2}^+$	0.3

Then $G_1 \times G_2$ will be an IVFG defined by

	(u_1, v_1)	(u_1, v_2)	(u_2, v_1)	(u_2, v_2)
$\mu_{A_1}^- \times \mu_{A_2}^-$	0.4	0.4	0.4	0.4
$\mu_{A_1}^+ \times \mu_{A_2}^+$	0.5	0.5	0.5	0.5

	$(u_1, v_1)(u_1, v_2)$	$(u_1, v_1)(u_2, v_1)$
$\mu_{B_1}^- \times \mu_{B_2}^-$	0.2	0.3
$\mu_{B_1}^+ \times \mu_{B_2}^+$	0.3	0.4
	$(u_1, v_2)(u_2, v_2)$	$(u_2, v_1)(u_2, v_2)$
$\mu_{B_1}^- \times \mu_{B_2}^-$	0.3	0.2
$\mu_{B_1}^+ \times \mu_{B_2}^+$	0.4	0.3

Using routine computations, we can see that both $G_1 \times G_2$ and G_1 are TRIVFGs, but G_2 is not a TRIVFG.

Now we proceed to obtain some necessary and sufficient conditions for the Cartesian product of two TRIVFGs to be totally regular under some restrictions

Theorem 4.1.

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \geq B_2$ and $A_2 \geq B_1$, and $rmin(A_1, A_2)$ is a constant, then $G_1 \times G_2$ is totally regular if and only if G_1 and G_2 are regular.

Proof:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. Suppose $A_1 \geq B_2$ and $A_2 \geq B_1$,

Then by theorem 3.1,

$$td_{G_1 \times G_2}^-(u_1, v_1) = d_{G_1}^-(u_1) + d_{G_2}^-(v_1) + min\left(\mu_{A_1}^-(u_1), \mu_{A_2}^-(v_1)\right)$$ and

$$td_{G_1 \times G_2}^+(u_1, v_1) = d_{G_1}^+(u_1) + d_{G_2}^+(v_1) + min\left(\mu_{A_1}^+(u_1), \mu_{A_2}^+(v_1)\right)$$

Now suppose that $min(A_1, A_2) = [c_1, c_2]$, a constant. Then,

$td_{G_1 \times G_2}^-(u_1, v_1) = d_{G_1}^-(u_1) + d_{G_2}^-(v_1) + c_1$ and

$td_{G_1 \times G_2}^+(u_1, v_1) = d_{G_1}^+(u_1) + d_{G_2}^+(v_1) + c_2$

Suppose G_1 and G_2 are regular IVFGs with degrees $[k_1, k_2]$ and $[l_1, l_2]$ respectively. Then the above equations become $td_{G_1 \times G_2}^-(u_1, v_1) = k_1 + l_1 + c_1$ and $td_{G_1 \times G_2}^+(u_1, v_1) = k_2 + l_2 + c_2$ where $(u_1, v_1) \in V_1 \times V_2$ is arbitrary which shows that $G_1 \times G_2$ is totally regular.

Conversely, suppose that $G_1 \times G_2$ is totally regular. We have to prove that G_1 and G_2 are regular. Then for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,

$td_{G_1 \times G_2}^-(u_1, v_1) = td_{G_1 \times G_2}^-(u_2, v_2)$

$\Longrightarrow d_{G_1}^-(u_1) + d_{G_2}^-(v_1) + c_1 =$

$d_{G_1}^-(u_2) + d_{G_2}^-(v_2) + c_1$ [By theorem 3.1]

$\Longrightarrow d_{G_1}^-(u_1) + d_{G_2}^-(v_1) = d_{G_1}^-(u_2) + d_{G_2}^-(v_2)$

Fix $u \in V_1$ and consider (u, v_1) and (u, v_2) in $V_1 \times V_2$, where $v_1, v_2 \in V_2$ are arbitrary. Then, $d_{G_1}^-(u) + d_{G_2}^-(v_1) = d_{G_1}^-(u) + d_{G_2}^-(v_2)$ which implies $d_{G_2}^-(v_1) = d_{G_2}^-(v_2)$.

Similarly, $d_{G_2}^+(v_1) = d_{G_2}^+(v_2)$. This is true for all $v_1, v_2 \in V_2$. Thus G_2 is a RIVFG.

Now fix $v \in V_2$ and consider (u_1, v) and (u_2, v) in $V_1 \times V_2$, where $u_1, u_2 \in V_1$ are arbitrary. Then, $d^-_{G_1}(u_1) + d^-_{G_2}(v) = d^-_{G_1}(u_2) + d^-_{G_2}(v)$ which implies $d^-_{G_1}(u_1) = d^-_{G_1}(u_2)$.
Similarly, $d^+_{G_1}(u_1) = d^+_{G_1}(u_2)$. This is true for all $u_1, u_2 \in V_1$. Thus G_1 is a RIVFG

Theorem 4.2.

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. If $A_1 \geq B_2$ and $A_2 \geq B_1$, and $rmax(A_1, A_2)$ is a constant, then $G_1 \times G_2$ is totally regular if and only if G_1 and G_2 are totally regular.

Proof:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs. Suppose $A_1 \geq B_2$ and $A_2 \geq B_1$, Then by theorem 3.2,

$$td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + td^-_{G_2}(v_1) - max\left(\mu^-_{A_1}(u_1), \mu^-_{A_2}(v_1)\right) \text{ and}$$

$$td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + td^+_{G_2}(v_1) - max\left(\mu^+_{A_1}(u_1), \mu^+_{A_2}(v_1)\right)$$

Now suppose that $rmax(A_1, A_2) = [c'_1, c'_2]$, a constant. Then $td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + td^-_{G_2}(v_1) - c'_1$ and $td^+_{G_1 \times G_2}(u_1, v_1) = d^+_{G_1}(u_1) + d^+_{G_2}(v_1) - c'_2$
Suppose G_1 and G_2 are TRIVFGs with degrees $[k'_1, k'_2]$ and $[l'_1, l'_2]$ respectively. Then the above equations become $td^-_{G_1 \times G_2}(u_1, v_1) = k'_1 + l'_1 - c'_1$ and $td^+_{G_1 \times G_2}(u_1, v_1) = k'_2 + l'_2 - c'_2$ which shows that $G_1 \times G_2$ is totally regular.

Conversely, suppose that $G_1 \times G_2$ is totally regular. We have to prove that G_1 and G_2 are totally regular. Then for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$, $td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1 \times G_2}(u_2, v_2)$

$$\Rightarrow td^-_{G_1}(u_1) + t\,d^-_{G_2}(v_1) - c'_1 = td^-_{G_1}(u_2) + td^-_{G_2}(v_2) - c'_1 \text{[By theorem 3.2]}$$

$$\Rightarrow td^-_{G_1}(u_1) + td^-_{G_2}(v_1) = td^-_{G_1}(u_2) + t\,d^-_{G_2}(v_2)$$

Fix $u \in V_1$ and consider (u, v_1) and (u, v_2) in $V_1 \times V_2$, where $v_1, v_2 \in V_2$ are arbitrary. Then, $td^-_{G_1}(u) + td^-_{G_2}(v_1) = td^-_{G_1}(u) + td^-_{G_2}(v_2)$ which implies $td^-_{G_2}(v_1) = td^-_{G_2}(v_2)$.
Similarly, $td^+_{G_2}(v_1) = td^+_{G_2}(v_2)$.This is true for all $v_1, v_2 \in V_2$. Thus G_2 is a TRIVFG.
Now fix $v \in V_2$ and consider (u_1, v) and (u_2, v) in $V_1 \times V_2$, where $u_1, u_2 \in V_1$ are arbitrary. Then, $td^-_{G_1}(u_1) + t\,d^-_{G_2}(v) = td^-_{G_1}(u_2) + td^-_{G_2}(v)$ which implies $td^-_{G_1}(u_1) = td^-_{G_1}(u_2)$.
Similarly, $td^+_{G_1}(u_1) = td^+_{G_1}(u_2)$.This is true for all $u_1, u_2 \in V_1$. Thus G_1 is a TRIVFG

Theorem 4.3.

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $A_1 \leq B_2$ and A_1 is a constant function. Then $G_1 \times G_2$ is totally regular if and only if G_1 is regular and G_2 is partially regular.

Proof:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $A_1 \leq B_2$. Then, by theorem 3.3,

$$td^-_{G_1 \times G_2}(u_1, v_1) = d^-_{G_1}(u_1) + \mu^-_{A_1}(u_1) dG^*_2(v_1) + \mu^-_{A_1}(u_1) \text{ and}$$

$$td^+_{G_1 \times G_2}(u_1, v_1) = d^+_{G_1}(u_1) + \mu^+_{A_1}(u_1) dG^*_2(v_1) + \mu^+_{A_1}(u_1)$$

Let $A_1 = [c_1, c_2]$, a constant. Then the above equations become $td^-_{G_1 \times G_2}(u_1, v_1) = d^-_{G_1}(u_1) + c_1 dG^*_2(v_1) + c_1$ and $td^+_{G_1 \times G_2}(u_1, v_1) = d^+_{G_1}(u_1) + c_2 dG^*_2(v_1) + c_2$.
Suppose G_1 is regular with degree $[k_1, k_2]$ and G_2 is partially regular with degree m. Then the above equations become $td^-_{G_1 \times G_2}(u_1, v_1) = k_1 + c_1 m + c_1 = k_1 + (m+1)c_1$ and $td^+_{G_1 \times G_2}(u_1, v_1) = k_2 + c_2 m + c_2 = k_2 + (m+1)c_2$ which shows that $G_1 \times G_2$ is totally regular.

Conversely, suppose that $G_1 \times G_2$ is totally regular. We have to prove that G_1 is regular and G_2 is partially regular. Then for any two points (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$, $td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1 \times G_2}(u_2, v_2)$

$$\Rightarrow d^-_{G_1}(u_1) + c_1 dG^*_2(v_1) + c_1 = d^-_{G_1}(u_2) + c_1 dG^*_2(v_2) + c_1 \text{[By theorem 3.3]}$$

$$\Rightarrow d^-_{G_1}(u_1) + c_1 dG^*_2(v_1) = d^-_{G_1}(u_2) + c_1 dG^*_2(v_2)$$

Fix $u \in V_1$ and consider (u, v_1) and (u, v_2) in $V_1 \times V_2$, where $v_1, v_2 \in V_2$ are arbitrary. Then, $d^-_{G_1}(u) + c_1 dG^*_2(v_1) = d^-_{G_1}(u) + c_1 dG^*_2(v_2)$ which implies $dG^*_2(v_1) = dG^*_2(v_2)$ .This is true for all $v_1, v_2 \in V_2$. Thus G^*_2 is regular. Hence G_2 is a PRIVFG.
Now fix $v \in V_2$ and consider (u_1, v) and (u_2, v) in $V_1 \times V_2$, where $u_1, u_2 \in V_1$ are arbitrary. Then, $d^-_{G_1}(u_1) + c_1 dG^*_2(v) = d^-_{G_1}(u_2) + c_1 dG^*_2(v)$ which implies $d^-_{G_1}(u_1) = d^-_{G_1}(u_2)$.
Similarly, $d^+_{G_1}(u_1) = d^+_{G_1}(u_2)$.This is true for all $u_1, u_2 \in V_1$. Thus G_1 is a RIVFG

Theorem 4.4.

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $A_1 \leq B_2$ and A_1 is a constant function. Then $G_1 \times G_2$ is totally regular if and only if G_1 is totally regular and G_2 is partially regular.

Proof:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $A_1 \leq B_2$.
Then, by theorem 3.4,

$$td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + \mu^-_{A_1}(u_1) dG^*_2(v_1)$$

and

$$td^+_{G_1 \times G_2}(u_1, v_1) = td^+_{G_1}(u_1) + \mu^+_{A_1}(u_1) dG^*_2(v_1) .\ .$$

Let $A_1 = [c_1, c_2]$, a constant . Then the above equations become
$td^-_{G_1 \times G_2}(u_1, v_1) = td^-_{G_1}(u_1) + c_1 dG^*_2(v_1)$ and

$td^{+}_{G_1 \times G_2}(u_1, v_1) = td^{+}_{G_1}(u_1) + c_2 dG^{*}_2(v_1)$.

Suppose G_1 is totally regular with degree $[k'_1, k'_2]$ and G_2 is partially regular with degree m. Then the above equations become $td^{-}_{G_1 \times G_2}(u_1, v_1) = k'_1 + c_1 m$ and $td^{+}_{G_1 \times G_2}(u_1, v_1) = k'_2 + c_2 m$ which shows that $G_1 \times G_2$ is totally regular.

Conversely, suppose that $G_1 \times G_2$ is totally regular. We have to prove that G_1 is totally regular and G_2 is partially regular. Then for any two points (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,
$td^{-}_{G_1 \times G_2}(u_1, v_1) = td^{-}_{G_1 \times G_2}(u_2, v_2)$
$\Longrightarrow td^{-}_{G_1}(u_1) + c_1 dG^{*}_2(v_1) =$
$td^{-}_{G_1}(u_2) + c_1 dG^{*}_2(v_2)$[By theorem 3.4]

Fix $u \in V_1$ and consider (u, v_1) and (u, v_2) in $V_1 \times V_2$, where $v_1, v_2 \in V_2$ are arbitrary. Then,
$td^{-}_{G_1}(u) + c_1 dG^{*}_2(v_1) = td^{-}_{G_1}(u) + c_1 dG^{*}_2(v_2)$
which implies $dG^{*}_2(v_1) = dG^{*}_2(v_2)$ .This is true for all $v_1, v_2 \in V_2$. Thus G^{*}_2 is regular. Hence G_2 is a PRIVFG.

Now fix $v \in V_2$ and consider (u_1, v) and (u_2, v) in $V_1 \times V_2$, where $u_1, u_2 \in V_1$ are arbitrary. Then,
$td^{-}_{G_1}(u_1) + c_1 dG^{*}_2(v) = td^{-}_{G_1}(u_2) + c_1 dG^{*}_2(v)$
which implies $td^{-}_{G_1}(u_1) = td^{-}_{G_1}(u_2)$. Similarly,
$td^{+}_{G_1}(u_1) = td^{+}_{G_1}(u_2)$. This is true for all $u_1, u_2 \in V_1$. Thus G_1 is a TRIVFG

V. CONCLUSION

Cartesian Product of graphs have applications in many branches like coding theory, network designs, chemical graph theory etc. In this paper, we have obtained the total degree of a vertex in the Cartesian Product of two IVFGs in terms of degree and total degree of vertices of component graphs. This will be very helpful in analyzing many properties of Cartesian Product of IVFGs. We have observed that the Cartesian Product of two TRIVFGs need not be a TRIVFG. Also we derived some necessary and sufficient conditions for the Cartesian product of two TRIVFGs to be totally regular under some restrictions.

ACKNOWLEDGEMENT

I express my sincere gratitude to the University Grants Commission for granting me FDP leave for completing my Ph. D work. I am also extremely grateful to the reviewers and the Editor-in-Chief for their valuable comments and suggestions for improving the paper

REFERENCES

[1] M.Akram, "Interval – valued fuzzy line graphs", *Neural Computing applications*, vol. 21 , pp. 145-150, 2012.

[2] M.Akram , N.O.Alsheri and W.A.Dudek, "Certain Types of Interval – Valued Fuzzy Graphs", *Journal of Applied Mathematics*, 2013.

[3] M.Akram and W.A.Dudek, "Interval – valued fuzzy graphs", *Computers and Mathematics with Applications*, vol. 61, pp. 289-299, 2011.

[4] M.Akram, M.Murthaza Yousaf and W.A.Dudek," Self centered interval – valued fuzzy graphs", *Africa Mathematika* vol. 26, pp. 887-898, September 2015.

[5] Ann Mary Philip, "Interval – valued Fuzzy Bridges and Interval – valued Fuzzy Cutnodes", *Annals of Pure and Applied Mathematics*, Vol.14, No.3, pp. 473 – 487, 2017.

[6] Basheer Ahamed Mohideen," Strong and regular interval-valued fuzzy graphs", *Journal of Fuzzy Set Valued Analysis* , No.3,pp. 215-223, 2015.

[7] P. Bhattacharya, "Some remarks on fuzzy graphs", *Pattern Recognition letters, vol.* 6 , pp. 297-302, 1987.

[8] K.R.Bhutani, "On automorphisms of fuzzy graphs", *Pattern Recognition letters* vol, 9 , pp. 159-162,1989.

[9] K.R.Bhutani and A.Battou, "On M-strong fuzzy graphs", *Information Sciences*, vol. 155 , pp. 103-109,2003.

[10] K.R.Bhutani and A.Rosenfeld, "Geodesics in fuzzy graphs", *Electronic Notes in Discrete Mathematics*, vol. 15, pp. 51-54, 2003.

[11] K.R.Bhutani and A.Rosenfeld, "Fuzzy end nodes in fuzzy graphs", *Information Sciences*, vol. 152 , pp. 323-326,2003.

[12] K.R.Bhutani and A.Rosenfeld,"Strong arcs in fuzzy graphs", *Information Sciences, vol.* 152 , pp. 319-322, 2003.

[13] J.Hongmei,W.Lianhua,"Interval-valued fuzzy subsemi-groups and subgroups associated by interval-valued fuzzy graphs"; *2009 WRI Global Congress on Intelligent Systems*, pp. 484- 487, 2009.

[14] Kaufman, A.,"Introduction a la Theorie des Sous-emsembles Flous", Masson et Cie , vol. 1, 1973.

[15] J.N.Mordeson, "Fuzzy line graphs", *Pattern Recognition letters* , vol. 14 , pp. 381-384, 1993.

[16] J.N.Mordeson and P.S.Nair, "Cycles and cocycles of fuzzy graphs", *Information Sciences, vol.* 90 , pp. 39-49, 1996.

[17] J.N.Mordeson and P.S.Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica-verlag, Heidelberg, 2000

[18] J.N.Mordeson and C.S.Peng, "Operations on fuzzy graphs", *Information Sciences, vol.* 79, pp. 159-170, 1994.

[19] A. Nagoor Gani and M. Basheer Ahmed, "Order and Size in Fuzzy Graph", *Bulletin of Pure and Applied Sciences*, Vol.22E, No.1, pp. 145-148, January 2003.

[20] A .Nagoor Gani and J.Malarvizhi, "Isomorphism on Fuzzy graphs", *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering* , Vol. 2, No. 11, 2008.

[21] A. Nagoor Gani and K. Radha, "Some sequences in fuzzy graphs", Far East Journal of Applied Mathematics, vol. 31, No. 3, pp. 321-325, June 2008.

[22] A. Nagoor Gani and K. Radha, "On Regular Fuzzy Graphs", Journal of Physical Sciences, Vol 12, pp. 33-40, 2008.

[23] A. Nagoor Gani and H. Sheik Mujibur Rahman, "Totally Regular Property of Cartesian Product of Intuitionistic Fuzzy Graphs", *Advances in Fuzzy Mathematics*, vol. 12, No.1, pp. 63-76, 2017.

[24] A. Nagoor Gani and H. Sheik Mujibur Rahman,, "Total Degree of a Vertex in Cartesian Product and Composition of Some Intuitionistic Fuzzy Graphs", International Journal of Fuzzy Mathematical Archive, vol. 9, No. 2, pp. 135 – 143, 2015.

[25] M. Pal, and H. Rashmanlou, "Irregular interval-valued fuzzy graphs", *Annals of Pure and Applied Mathematics*, Vol.3, No.1,pp. 56-66, 2013

[26] M. Pal, S. Samanta and H. Rashmanlou, "Some results on interval-valued fuzzy graphs"; *International Journal of Computer Science and Electronics Engineering*, Vol.3, Issue 3, pp. 205-211, 2015.

[27] Pradip Debnath, "Domination in interval-valued fuzzy graphs", *Annals of Fuzzy Mathematics and Informatics* ,2013.

[28] H. Rashmanlou, Y. B. Jun, "Complete interval-valued fuzzy graphs", *Annals of Fuzzy Mathematics and Informatics*, vol. 6. No. 3, pp. 677-687, 2013.

[29] H. Rashmanlou and M. Pal," Balanced interval-valued fuzzy graphs", *Journal of Physical Sciences*, vol 17, pp. 43-57, December 2013.

[30] H. Rashmanlou and M. Pal, "Antipodal interval-valued fuzzy graphs", *International Journal of Applications of*

Fuzzy Sets and Artificial Intelligence, Vol.3 , pp. 107-130, 2013.

[31] H. Rashmanlou and M. Pal, "Isometry on interval-valued fuzzy graphs", *International Journal on Fuzzy Mathematical Archive*, Vol.3, pp. 28-35, 2013.

[32] H. Rashmanlou and M. Pal, "Some properties of Highly Irregular interval-valued fuzzy graphs"; *World Applied Sciences Journal*, vol. 27, No. 12 , pp. 1756-1773, 2013.

[33] S. Ravi Narayanan and N.R. Santhi Maheswari; "Strongly Edge Irregular interval-valued fuzzy graphs", *International Journal of Mathematical Archive*, vol. 7, No.1, pp. 192-199, 2016.

[34] Rosenfeld A.,"Fuzzy Graphs, *Fuzzy Sets and their Applications". In: Zadeh, L.A., Fu, K.S., Shimura, M.(eds.),* Academic Press,New York ,pp. 77-95, 1975.

[35] Souriar Sebastian and Ann Mary Philip, "On total regularity of the join of two interval valued fuzzy graphs", *International Journal of Scientific and Research Publications,* Vol. 6,Issue 12,pp. 45-55, December 2016.

[36] Souriar Sebastian and Ann Mary Philip, "Regular and Edge regular interval valued fuzzy graph"s, *Journal of Computer and Mathematical Sciences,* Vol 8, No. 7*, pp. 309-322, July 2017*.

[37] M.S. Sunitha and A. Vijayakumar, "Complement of a fuzy graph", *Indian Journal of Pure and Applied Mathematics*, vol. 33, No. 9, pp. 1451-1464, September 2002.

[38] M.S. Sunitha and A. Vijayakumar, "A Characterization of fuzzy trees", *Information Sciences*, vol. 113, pp. 293 – 300, 1999.

[39] M. S. Sunitha and A. Vijayakumar, "Blocks in fuzzy graphs", *The Journal of Fuzzy Mathematics, vol.* 13 , No.1, pp. 13-23, 2005.

[40] M. S. Sunitha and A. Vijayakumar, "Some Metric Aspects of Fuzzy Graphs", *Proceedings of the conference on Graph connections, Cochin University of Science & Tech*, Cochin, India, pp. 111-114, January 1998.

[41] A. A. Talebi and H. Rashmanlou, "Isomorphism on interval-valued fuzzy graphs", *Annals of Fuzzy Mathematics and Informatics, vol.* 6, No. 1 , pp. 47-58, 2013.

[42] A. A. Talebi, H. Rashmanlou and Reza Ameri, "New Concepts of Product interval-valued fuzzy graphs", *Journal of Applied Mathematics and Informatics*,Vol.34 ,No.3-4, pp. 179-192, 2016.

[43] L.A.Zadeh,"FuzzySet", *Information Control*, vol. 8, pp. 338 – 353, 1965.

[44] L.A.Zadeh, "The concept of a Linguistic variables and its application to approximate reasoning", *Information Science* vol. 8, pp. 199 – 249, 1975.

.