Performance Analysis of Three Stage Tandem Queues

Sreekanth. K[1] and Kamlesh Kumar[2]

Department of Mathematics, Central University

of Jammu, Jammu-180 011, India.

Abstract: *This paper deals with the three stage tandem queues and performance analysis is rendered through analytic approach. After constructing the system governing equations and using the Burke's theorem, we determine the steady-state probabilities and various performance measures of the three stage tandem queueing system.*

Keywords: Tandem Queue, Burkes Theorem and Performance Analysis.

1 Introduction

Over the past few decades tandem queueing system has been studied extensively because of its widespread prevalence in our real life situations. A tandem queue is a queueing system organized in series queue wherein service facilities are given in a succession and the flow of customers is always in a single path from facility to facility or in a series. As far as the application concern the study of tandem queueing system has found a great deal of attention in the areas such as computer systems, tollbooths, communication networks, production lines, etc.

Nowadays, Tandem queueing model has drawn a good deal of attention for many researchers because of it's practical applications in real world. Hunt[8] studied a finite server tandem queueing system and discussed the notion of approximate decomposition. Burke[4] has considered a queueing system with the poisson arrival flow and exponential distributed service times, he showed that the output process also follow poisson with same parameter as

in input. Tsiolras[14] has analysed a two-station tandem queue with general arrivals and exponential services. He has used recursive method approach to get the closed form expressions for the conditional expected queueing times at the first station of tanden queue. For analyzing the performance indices of a finite closed tandem queueuing system Bouhchouch et al.[3]have proposed an approximation method. Chand and Chen[5]have proposed an admission control policy by studying a two-stage no-wait tandem queueing system. Li et al.[11] have discussed a finite buffers tandem queue. By using the idea of critical paths they have presented a general optimization procedure with the PA approach. Song and Mehmet Ali[13] have studied tandem queues with Markovian modeling, in addition to this they have obtained the mean, variance of queue length and mean packet-delay in the tandem queueing system. They also determined the probability generating function (PGF) method for obtaining the queue length distribution. Kim et al.[10]have studied a two phase tandem retrial queues with a batch Markovian arrival process and have obtained various analytic results. By employing a fuzzy simulation algorithm Azadeh et al.[1]have examined the various results with the optimization of a tandem queue system. Mohammad et al.[12] have considered the server allocating problem with no buffers tandem queueing system. By considering arrival flow Markovian and service time generalized phase-type distribution Kim et al.[9] have studied a tandem queueuing system with heterogeneous customers. They calculated various performance in-

dices and also derived ergodicity condition. Using matrix-analytic method He and Chao[7] have analyzed a tandem queues with heterogeneous servers. They evaluated explicit results for various performance indices. Do[6] has taken into consideration a tandem queueing system with two heterogeneous servers where arrival is poisson and service times follows exponential. Also he obtained a closed form solution. Baumann and Sandmann[2] have examined the performance analysis of multiserver tandem queues in which arrival follows Markovian and service times is phase-type.

The remaining part of the paper have been divided into four sections. Section 2 presents the model in description. Section 3 provides the steady state solutions to the tandem queueing system governing equations by using the Burke's theorem. Section 4 comprise the various performance indices of the three stage tandem queueing system. In the last, the concluding remark is made in section 5.

2   Model Description

A tandem queue with three service stations is taken into consideration wherein with a Poisson rate λ the customers from outside enter to the station S_1. After completing the service at the station S_1, the customer will have to go to station S_2 from the station S_2 for receiving service, in front of the station S_3 they join queue. The Customers leave the queueing system only after the completion of service at station S_3. The capacity of the queue space is presumed to be infinite and at stations S_1, S_2 and S_3 service time follows exponential distributions with parameters μ_1, μ_2 and μ_3 respectively. It is also to be noted that at a time one customer is able to avail of the service from each station.
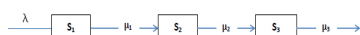

Figure 1: Three - Stage Tandem Queue

In order to evaluate the steady-state probability $P_{k,l,m}$ of k customers at S_1 and l and m customers respectively at stations S_2 and S_3. where k, l, m $\geq$ 0.

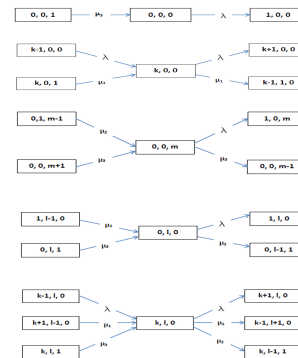Using the state-transition diagrams we can write the balance equations (fig.2 and fig.3).

Figure 2: State Transition Diagram

It is only by means of the state diagram that from state $(0,0,1)$ only the state $(0,0,0)$ can be reached. The state $(1,0,0)$ is the only state into which the central state $(0,0,0)$ can go and thus the central state can go to no other state except the state $(1,0,0)$. Likewise from the states $(m-1,0,0)$ and $(m,0,1)$ only the state $(m,0,0)$ can be attained. The customers can go either to the state $(m+1,0,0)$ or to the state $(m-1,1,0)$ only from the state $(m,0,0)$. The below mentioned Figure demonstrates similar arguments which hold good for other states too.
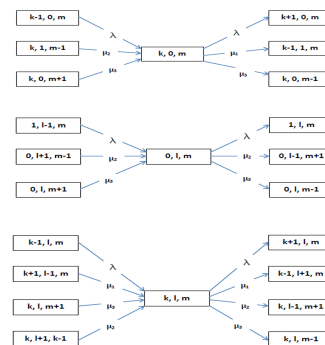

Figure 3: State Transition Diagram

Notations The subsequent notations are used to formulate the tandem queueing model with three stations S_1, S_2, S_3. The steady state probabilities $P_{k,l,m}$ define that there are k number of customers at station S_1 , l number of customers in S_2 and m customers in S_3. State of the model is denoted by

(k, l, m) and thus the possible states of the queueing systems are

P_k – Probability that k customers at the station S_1

P_l – Probability that l customers at the station S_2

P_m – Probability that m customers at the station S_3

$P_{0,0,0}$ – Probability that no customer at any station S_1, S_2 and S_3.

$P_{k,0,0}$ – The probability that k customers at station S_1 and no customers at stations S_2 and S_3.

$P_{0,l,0}$ – Probability that no customer at station S_1 and S_3, but l customers are at station S_2.

$P_{0,0,m}$ – Probability that no customer at station S_1 and S_2, but m customers are at station S_3.

$P_{k,l,0}$ – Probability that k customers at station S_1 and l customers at station S_2 no customer at station S_3.

$P_{k,0,m}$ – Probability k customers are at station S_1 and no customer at station S_2 and also m customers at station S_3.

$P_{0,l,m}$ – Probability that no customer at station S_1, but l customers at station S_2 and m customers at station S_3.

$P_{k,l,m}$ – Probability that k customers at station S_1, l customers at station S_2 and m customers at S_3.

3   Governing Equations

By using the transition state diagram in figure 2 and figure 3, we formulate the steady state balance equations as below;

$$\lambda P_{0,0,0} = \mu_3 P_{0,0,1} \tag{1}$$

$$(\lambda + \mu_3)P_{0,0,m} = \lambda P_{0,1,m-1} + \mu_1 P_{0,0,l+1} \tag{2}$$

$$(\lambda + \mu_1)P_{k,0,0} = \lambda P_{k-1,0,0} + \mu_3 P_{k,0,1} \tag{3}$$

$$(\lambda + \mu_2)P_{0,l,0} = \mu_1 P_{1,l-1,0} + \mu_3 P_{0,l,1} \tag{4}$$

$$(\lambda + \mu_1 + \mu_2)P_{k,l,0} = \lambda P_{k-1,l,0} + \mu_1 P_{k+1,l-l,0}$$
$$+ \mu_3 P_{k,l,1} \tag{5}$$

$$(\lambda + \mu_1 + \mu_3)P_{k,0,m} = \lambda P_{k-1,0,m} + \mu_2 P_{k,1,m-1}$$
$$+ \mu_3 P_{k,0,m+1} \tag{6}$$

$$(\lambda + \mu_2 + \mu_3)P_{0,l,m} = \mu_1 P_{1,l-1,m} + \mu_2 P_{0,l+1,m-1}$$
$$+ \mu_3 P_{0,l,m+1} \tag{7}$$

$$(\lambda + \mu_1 + \mu_2 + \mu_3)P_{k,l,m} = \lambda P_{k-1,l,m} + \mu_1 P_{k+1,l-1,m}$$
$$+ \mu_3 P_{k,l,m+1} + \mu_2 P_{k,l+1,m-1} \tag{8}$$

Moreover, we know that the sum of all probabilities must equal to one. i.e.,

$$\sum_k \sum_l \sum_m P_{k,l,m} = 1 \tag{9}$$

In this case, the solution of the above governing equations are solved by making the use of Burke's theorem. The statement of the theorem is given below:

Burke's Theorem[4]

"For a queueing system with Poisson input, a single waiting line without defections, and identically distributed independent(negative) exponential service times, the equilibrium distribution of the number of service completions in an arbitrary time interval is same as the input distribution, for any number of servers".

In order to demonstrate the result, Burke made the suppositions that the length of the average interarrival $\frac{1}{\lambda}$, there are s number of servers all having an exponentially distributed independent service times with average $\frac{1}{\mu}$ and $\mu s > \lambda$. By making use of the assumptions instead of proving the equilibrium distribution of the number of customers finishing service between an arbitrary interval of length T follow poisson with parameter λT , Burke proved an analogous result, that the time intervals among subsequent service completions are independent, exponentially distributed which is same as the interarrival times.(see[4])

The arrival process to S_2, which is just like the departure process from S_1, is also poisson with parameter λ from the theorem. In case of the servers

S_2 and S_3 same is held. Hence, the queueing systems are $M/M/1$ queueing model. As far as $M/M/1$ queueing model is concerned, now we determine the probability distribution for the three stage tandem queue by using the Burke's theorem.

P_k- Probability that k number of customers at the station S_1

$$= \left[\frac{\lambda}{\mu_1}\right]^k \left[1 - \frac{\lambda}{\mu_1}\right] \qquad (10)$$

P_l- Probability that l number of customers at the station S_2

$$= \left[\frac{\lambda}{\mu_2}\right]^l \left[1 - \frac{\lambda}{\mu_2}\right] \qquad (11)$$

P_m- Probability that m number of customers at the station S_3

$$= \left[\frac{\lambda}{\mu_3}\right]^m \left[1 - \frac{\lambda}{\mu_3}\right] \qquad (12)$$

At each station S_1, S_2, S_3 the number of customers are independent random variables, thus the probability of k number of customers at station S_1 and l number of customers at station S_2 and m customers at station S_3 are jointly given by,

$$P_{k,l,m} = \left[\frac{\lambda}{\mu_1}\right]^k \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_2}\right]^l \left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_3}\right]^m \left[1 - \frac{\lambda}{\mu_3}\right] \qquad (13)$$

$$\text{where k, } l, \text{ m} \geq 0$$

4 Performance Measures

For examining the series queueing system behavior, various performance indices are constructed using steady state probabilities distributions as-

(i) The average number of customers in the system

$$L_s = \sum_k kP_k + \sum_l lP_l + \sum_m mP_m$$

$$= \sum_{k=0}^{\infty} k\left[\frac{\lambda}{\mu_1}\right]^k \left[1 - \frac{\lambda}{\mu_1}\right] + \sum_{l=0}^{\infty} l\left[\frac{\lambda}{\mu_2}\right]^l \left[1 - \frac{\lambda}{\mu_2}\right]$$

$$+ \sum_{m=0}^{\infty} m\left[\frac{\lambda}{\mu_3}\right]^m \left[1 - \frac{\lambda}{\mu_3}\right]$$

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda} + \frac{\lambda}{\mu_3 - \lambda}$$

(ii) The average waiting time of the customer in the system

$$W_s = \frac{1}{\lambda}L_s$$

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} + \frac{1}{\mu_3 - \lambda}$$

(iii) The probability that the three service stations are idle

$$P_{0,0,0} = \left[\frac{\lambda}{\mu_1}\right]^0 \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_2}\right]^0 \left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_3}\right]^0 \left[1 - \frac{\lambda}{\mu_3}\right]$$

$$= \left[\frac{\mu_1 - \lambda}{\mu_1}\right]\left[\frac{\mu_2 - \lambda}{\mu_2}\right]\left[\frac{\mu_3 - \lambda}{\mu_3}\right]$$

(iv) The probability that k customers at station S_1 and no customers at S_2 and S_3

$$P_{k,0,0} = \left[\frac{\lambda}{\mu_1}\right]^k \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_2}\right]^0 \left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_3}\right]^0 \left[1 - \frac{\lambda}{\mu_3}\right]$$

$$= \lambda^k \left[\frac{\mu_1 - \lambda}{\mu_1^{k+1}}\right]\left[\frac{\mu_2 - \lambda}{\mu_2}\right]\left[\frac{\mu_3 - \lambda}{\mu_3}\right]$$

(v) The probability that no customers at stations S_1 and S_3 and l number of customers at station S_2

$$P_{0,l,0} = \left[\frac{\lambda}{\mu_1}\right]^0 \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_2}\right]^l \left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_3}\right]^0 \left[1 - \frac{\lambda}{\mu_3}\right]$$

$$= \lambda^l \left[\frac{\mu_1 - \lambda}{\mu_1}\right]\left[\frac{\mu_2 - \lambda}{\mu_2^{l+1}}\right]\left[\frac{\mu_3 - \lambda}{\mu_3}\right]$$

(vi) The probability that zero customers at stations S_1 and S_2 and m customers at station S_3

$$P_{0,0,m} = \left[\frac{\lambda}{\mu_1}\right]^0 \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_2}\right]^0 \left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_3}\right]^m \left[1 - \frac{\lambda}{\mu_3}\right]$$

$$= \lambda^m \left[\frac{\mu_1 - \lambda}{\mu_1}\right]\left[\frac{\mu_2 - \lambda}{\mu_2}\right]\left[\frac{\mu_3 - \lambda}{\mu_3^{m+1}}\right]$$

(vii) Probability that k, m customers at stations S_1 and S_2 respectively and no customers at station S_3

$$P_{k,0,m} = \lambda^{k+m} \left[\frac{\mu_1 - \lambda}{\mu_1^{k+1}}\right]\left[\frac{\mu_2 - \lambda}{\mu_2}\right]\left[\frac{\mu_3 - \lambda}{\mu_3^{m+1}}\right]$$

(viii) Probability that k, l, zero customers at stations S_1, S_2 S_3 respectively

$$P_{k,l,0} = \lambda^{k+l} \left[\frac{\mu_1 - \lambda}{\mu_1^{k+1}}\right] \left[\frac{\mu_2 - \lambda}{\mu_2^{l+1}}\right] \left[\frac{\mu_3 - \lambda}{\mu_3}\right]$$

(ix) Probability that 0, l, m customers at stations S_1, S_2 S_3 respectively

$$P_{0,l,m} = \lambda^{l+m} \left[\frac{\mu_1 - \lambda}{\mu_1}\right] \left[\frac{\mu_2 - \lambda}{\mu_2^{l+1}}\right] \left[\frac{\mu_3 - \lambda}{\mu_3^{m+1}}\right]$$

(x) Probability that k, l, m customers at stations S_1, S_2 S_3 respectively

$$P_{k,l,m} = \lambda^{k+l+m} \left[\frac{\mu_1 - \lambda}{\mu_1^{k+1}}\right] \left[\frac{\mu_2 - \lambda}{\mu_2^{l+1}}\right] \left[\frac{\mu_3 - \lambda}{\mu_3^{m+1}}\right]$$

(xi) At the stations S_1, S_2 and S_3 the probability that the customers excel the number k, l, m

$$
\begin{aligned}
P_{h>k,i>l,j>m} &= \sum_{h=k+1}^{\infty} P_h \cdot \sum_{i=l+1}^{\infty} P_i \cdot \sum_{j=m+1}^{\infty} P_j \\
&= \left[\frac{\lambda}{\mu_1}\right]^{k+1} \cdot \left[\frac{\lambda}{\mu_2}\right]^{l+1} \cdot \left[\frac{\lambda}{\mu_3}\right]^{m+1} \\
&= \frac{\lambda^{k+l+m+3}}{\mu_1^{k+1} \cdot \mu_2^{l+1} \cdot \mu_3^{m+1}}
\end{aligned}
$$

(xii) The average number of customers in the queue

$$
\begin{aligned}
L_q &= \sum_{k=1}^{\infty} (k-1)P_k + \sum_{l=1}^{\infty} (i-1)P_i + \sum_{m=1}^{\infty} (m-1)P_m \\
&= \frac{\lambda}{\mu_1} \cdot \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2} \cdot \frac{\lambda}{\mu_2 - \lambda} + \frac{\lambda}{\mu_3} \cdot \frac{\lambda}{\mu_3 - \lambda} \\
&= \lambda^2 \left[\frac{1}{\mu_1(\mu_1 - \lambda)} + \frac{1}{\mu_2(\mu_2 - \lambda)} + \frac{1}{\mu_3(\mu_3 - \lambda)}\right]
\end{aligned}
$$

(xiii) The average waiting time of a customer in the queue

$$
\begin{aligned}
W_q &= \frac{L_q}{\lambda} \\
W_q &= \lambda \left[\frac{1}{\mu_1(\mu_1 - \lambda)} + \frac{1}{\mu_2(\mu_2 - \lambda)} + \frac{1}{\mu_3(\mu_3 - \lambda)}\right]
\end{aligned}
$$

5 Conclusions

In this paper, the analysis of a tandem queue with three service stations has been done. By using the Burkes theorem we established the formulae for various performance measures. We can further extend this paper by increasing the number of service stations for the customers to study the Tandem queues. Also, for solving such tandem queue of multiple service stations, we will have to apply some numerical solutions approach by using various methods such as Matrix method, Successive over Relaxation method, Runge-Kutta's method and many more.

References

[1] Azadeh, A., Mohammad Ebrahima, R., Eivazy, H. : Parameter optimization of tandem queue systems with finite intermediate buffers via fuzzy simulation, *Performance Evaluation*, 67(2010) 353-360.

[2] Baumann, H. and Sandmann, W. : Multi-server tandem queue with Markovian arrival process, phase-type service times, and finite buffers, *European Journal of Operational Research*, 256(2017) 187-195.

[3] Bouhchouch, A., Frein, Y., Dallery, Y. : Performance evaluation of closed tandem queueing networks with finite buffers, *Performance Evaluation*, 26(1996) 115-132.

[4] Burke, P. J. : The output of a queueing system , *Operations Research*, 4(1956) 699-704.

[5] Chang, K, -H. and Chen, W, -F. : Admission control policies for two-stage tandem queues with no waiting spaces, *Computers and Operations Research*, 30(2003) 589-601.

[6] Do, T. V. : Aclosed-form solution for a toll booth tandem queue with two heterogeneous servers and exponential service times, *European Journal of Operational Research*, 247(2015) 672-675.

[7] He, Q, -M. and Chao, X. : A tollbooth tandem queue with heterogeneous servers, *European Journal of Operational Research*, 236(2014) 177-189.

[8] Hunt, G. C. : Sequential arrays of waiting lines, *Operations Research*, 4(1956) 674-683.

[9] Kim, C., Dudin, A., Dudina, O., Dudin, S. : Tandem queueing system with infinite and finite intermediate buffers and generalized phase-type service time distribution, *European Journal of Operational Research,* 235(2014) 170-179.

[10] Kim, C. S., Park, S. H., Dudin, A., Klimenok, V., Tsarenkov, G. : Investigaton of the $BMAP/G/1 \rightarrow \cdot/PH/1/M$ tandem queue with retrials and losses *Appl. Math. Modell,* 34(2010) 2926-2940.

[11] Li, Y., Cai, X., Tu, F., Shao, X. : Optimization of tandem queue systems with finite buffers, *Computers and Operations Research,* 31(2004) 963-984.

[12] Mohammad, H., Yarmand, Douglas, G., Down : Server allocation for zero buffer tandem queues, *European Journal of Operational Research,* 230(2013) 596-603.

[13] Song, X. and Mehmet Ali, M. : A performance analysis of discrete-time tandem queues with Markovian sources, *Performance Evaluation,* 66(2009) 524-543.

[14] Tsiotras, G. D. : Closed form expected queueing times of a tandem system with finite intermediate buffer: A recursive approach, *Appl. Math. Lett.,* 5(1992) 33-35.