Mappings Via ideals

Nitakshi Goyal
Department of Mathematics
Punjabi University,Patiala
Punjab, India.

December 8, 2017

Abstract

We will give characterizations of Pointwise-$\mathcal{I}$-continuous mapppings and Inversely-$\mathcal{I}$-open maps. We also give the relationship between them.

2010 Mathematics Subject Classification: 54C08, 54C10.
Keywords: inversely-$\mathcal{I}$-open, pointwise-$\mathcal{I}$-continuous, ideal.

1 Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski[2] and Vaidyanathaswamy[5]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X which satisfies that (i) $S \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $S \cup B \in \mathcal{I}$ and (ii) $S \in \mathcal{I}$ and $B \subset S$ implies $B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X known as ideal topological space and $(.)^* : \wp(X) \to \wp(X)$, called a local function[2] of S with respect to $\mathcal{I}$ and τ, is defined as follows: for $S \subseteq X$, $S^*(\mathcal{I}, \tau) = \{x \in X : U \cap S \notin \mathcal{I}$ for every open nhd. U of x in $X\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology, finer than τ, is defined by $cl^*(S) = S \cup S^*(\mathcal{I}, \tau)$[4]. When there is no chance of confusion, we will simply write S^* for $S^*(\mathcal{I}, \tau)$ and $\tau^*(\mathcal{I})$ for $\tau^*(\mathcal{I}, \tau)$.

Throughout this paper (X, τ) will denote topological space on which no separation axioms are assumed. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. For a subset S of X, $cl(S)$ and $int(S)$ will denote the closure of S, interior of S in (X, τ), respectively, $cl^*(S)$ and $int^*(S)$ will denote the closure of S, interior of S in (X, τ^*), respectively, and S^C will denote the complement of S in X.

We will also make use of the following results:

Lemma 1.1. *[3] For any sets X and Y, let $f : X \to Y$ be any map and S be any subset of X. Then:*

(a) $f^\#(S) = \{y \in Y : f^{-1}(y) \subseteq S\}$.

(b) $f^\#(S^C) = (f(S))^C$ *and so* $f^\#(S) = (f(S^C))^C$ *and* $f(S) = (f^\#(S^C))^C$.

(c) *f is onto if and only if* $f^\#(A) = f(A^\#)$.

(e) $E^\# = f^{-1}(f^\#(E))$.

Definition 1.1. [1] A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be pointwise-$\mathcal{I}$-continuous if the inverse image of every open set in Y is $\tau^*(\mathcal{I})$-open in X. Equivalently, $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is pointwise-$\mathcal{I}$-continuous if and only if $f : (X, \tau^*(\mathcal{I})) \to (Y, \sigma)$ is continuous.

2 Results

The following Theorems give characterization of pointwise-$\mathcal{I}$-continuous maps.

Theorem 2.1. *A map $g : (Y, \tau, \mathcal{I}) \to (Z, \sigma, \mathcal{J})$ with $\mathcal{J} = g(\mathcal{I})$ is pointwise-$\mathcal{I}$-continuous if and only if for each subset S of Y, $int(g^{\#}(S)) \subseteq g^{\#}(int^*(S))$.*

Proof. We know that g is pointwise-$\mathcal{I}$-continuous if and only if for each subset S of X, $g(cl^*(S)) \subseteq cl(g(S))$. But Lemma 1.1(b) implies that $g(cl^*(S)) \subseteq cl(g(S))$ if and only if $(g^{\#}(cl^*(S))^C)^C \subseteq cl(g^{\#}(S^C))^C = (int(g^{\#}(S^C)))^C$. Therefore, $g(cl^*(S)) \subseteq cl(g(S))$ if and only if $int(g^{\#}(S^C)) \subseteq g^{\#}(int^*(S^C))$. Since this equivalence holds for arbitrary subsets S of X, we have g is pointwise-$\mathcal{I}$-continuous if and only if for each subset S of Y, $int(g^{\#}(S)) \subseteq g^{\#}(int^*(S))$. □

Theorem 2.2. *Let $g : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ with $\mathcal{J} = g(\mathcal{I})$ be any surjective map. Then the following conditions are equivalent.*

(a) g is pointwise-$\mathcal{I}$-continuous.

(b) $int(g(S^{\#})) \subseteq g(int^*(S))^{\#})$ for all subsets S of X.

(c) $S^{\#}$ is τ^*-open in X whenever $g(S^{\#})$ is open in Y.

(d) for any saturated set S in X, S is τ^*-open in X whenever $g(S)$ is open in Y.

(e) for any saturated set S in X, S is τ^*-closed in X whenever $g(S)$ is closed in Y.

Proof. (a)⇔(b) follows using above theorem and Lemma 1.1(c).

(a)⇒(c): Let $g(S^{\#})$ is open in Y. So $g^{\#}(S)$ is open in Y using Lemma 1.1(c). Since g is pointwise-$\mathcal{I}$-continuous. Therefore, $g^{-1}(g^{\#}(S))$ is τ^*-open in X. Thus $S^{\#}$ is τ^*-open in X using Lemma 1.1(d).

(c)⇒(d): follows from (c) and follows from the fact that a subset S of X is saturated if and only if $S = S^{\#}$.

(d)⇒(e): Let $g(S)$ be closed in Y, where S is saturated in X. Then $(g(S))^C = g^{\#}(S^C)$ is open in Y using Lemma 1.1(b). So g is surjective implies $g((S^C)^{\#})$ is open in Y using Lemma 1.1(c). Slso S is saturated implies S^C is saturated and so $(S^C)^{\#} = S^C$. Therefore, $(g(S))^C = g(S^C)$ and so by (d), $g(S^C)$ is open in Y implies S^C is τ^*-open in X. Hence S is τ^*-closed in X.

(e)⇒(a): Let F be any closed subset of Y and let $S = g^{-1}(F)$. Then $g(S) = F$, since g is surjective, and so $g(S)$ is closed for a saturated subset S of X. Therefore, by condition (e), $S = g^{-1}(F)$ is τ^*-closed in Y. Hence g is pointwise-$\mathcal{I}$-continuous. □

Definition 2.1. A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be inversely-$\mathcal{I}$-open if for any subset A of X, $int(f(A)) \subseteq f(int^*(A))$.

The following theorem gives various characterizations of inversely-$\mathcal{I}$-open maps.

Theorem 2.3. *For any map $g : (Y, \tau, \mathcal{I}) \to (Z, \sigma)$, the following conditions are equivalent:*

(a) g is inversely-$\mathcal{I}$-open i.e. for each subset A of Y, $int(g(A)) \subseteq g(int^*(A))$.

(b) $g^{\#}(cl^*(A)) \subseteq cl(g^{\#}(A))$.

(c) if V is an open subset of Z and $V \subseteq g(Y)$, then each set consisting of exactly one point and so at least one point from each fiber $g^{-1}(y)$, where $y \in V$, is $\tau^*(\mathcal{I})$-open in Y.

(d) for any subset A of Y, A is $\tau^*(\mathcal{I})$-open in Y, whenever $g(A)$ is open subset of Z.

(e) for any subset A of Y, A is $\tau^*(\mathcal{I})$-closed in Y, whenever $g^{\#}(A)$ is closed subset of Z.

Proof. (a)⇔(b): It follows from the Lemma 1.1(b) and using $(cl(A))^C = int(A^C)$.

(a)⇒(c): Let g be inversely-$\mathcal{I}$-open and $V \subseteq g(Y)$ such that V is an open subset of Z. Let S be any set consisting of exactly one point from each fiber $g^{-1}(y)$, $y \in V$. We show that S is τ^*-open in Y. Now $g(S) = V$ and the image under g of any proper subset of S is a proper subset of V. If S is not τ^*-open in Y, then $int^*(S) \subset S$ but $int^*(S) \neq S$ and so $g(int^*(S)) \subset V$ but $g(int^*(S)) \neq V$. By (a) this implies that $int(g(S)) \subseteq g(int^*(S)) \subset V$ but $g(int^*(S)) \neq V$, contadicting the fact that $V = int(g(S))$, since V is open in Z. Hence (c) holds.

(c)⇒(d): Let $V = g(A)$ be an open subset of Z. We show that A is τ^*-open in Y. Consider the collection $\{S_y\}_y$, where each $S_y = g^{-1}(y) \cap A$, $y \in V$. Since $A = \cup_y S_y$ and each S_y contains at least one point from $g^{-1}(y)$, $y \in V$, the set A is τ^*-open in Y.

(d)⇒(e): For any set A in Y, let $g^{\#}(A)$ be closed in Z, then by Lemma 1.1(b), $(g(A^C))^C$ is closed in Z and so $g(A^C)$ is open in Z. Therefore, by (d), A^C is τ^*-open and so A is τ^*-closed in Y. Hence (e) holds.

(e)⇒(d): If $f(A)$ is open in Z, then by Lemma 1.1(b), $g^{\#}(A^C)$ is closed in Z. Therefore, by (e), A^C is τ^*-closed and so A is τ^*-open in Y. Hence (d) holds.

(d)⇒(a): Since for any set A in Y, $int(g(A)) \subseteq g(A)$, there exists $E \subseteq A$ such that $g(E) = int(g(A))$. Therefore, $g(E)$ is open subset of Z and by (d), E is τ^*-open in Y. Therefore, $E \subseteq int^*(A)$ and so $g(E) \subseteq g(int^*(A))$. Hence $int(g(A)) \subseteq g(int^*(A))$, i.e., g is inversely-$\mathcal{I}$-open. □

The following corollary gives the sufficient condition for a pointwise-$\mathcal{I}$-continuous map to be inversely-$\mathcal{I}$-open.

Corollary 2.1. *(a) Every pointwise-$\mathcal{I}$-continuous injective map is inversely-$\mathcal{I}$-open.*

(b) Let $f : (Y, \tau, \mathcal{I}) \to (Z, \sigma)$ be inversely-$\mathcal{I}$-open. Then f is pointwise-$\mathcal{I}$-continuous if $f(Y)$ is open in Z.

Proof. We prove only (b). Let G be open in Z. Then $f(f^{-1}(G)) = G \cap f(Y)$ is open in Z, since $f(Y)$ is open in Z. Then, $f^{-1}(G)$ is τ^*-open in Y by Theorem 2.3. Hence f is pointwise-$\mathcal{I}$-continuous. □

Next theorem gives different characterizations of pointwise-$\mathcal{I}$-closed maps (where a mapping $g : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ with $\mathcal{J} = f(\mathcal{I})$ is said to be pointwise-$\mathcal{I}$-closed if the image of every closed set is $\sigma^*(f(\mathcal{I}))$ - closed).

Theorem 2.4. *For any map $g : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ with $\mathcal{J} = g(\mathcal{I})$, the following conditions are equivalent.*

(a) g is pointwise-$\mathcal{I}$-closed;

(b) for each subset S of X, $cl^*(g(S)) \subseteq g(cl(S))$;

(c) for each subset S of X, $g^{\#}(int(S)) \subseteq int^*(g^{\#}(S))$;

(d) for each open subset G of X, $g^{\#}(G)$ is σ^*-open in Y.

Proof. (a)⇒(b): Let S be any subset of X. Then $g(S) \subseteq g(cl(S))$. So $cl^*(g(S)) \subseteq cl^*(g(cl(S)))$. But $cl(S)$ is closed subset of X, so by condition (a), $cl^*(g(cl(S))) = g(cl(S))$. Therefore, $cl^*(g(S)) \subseteq g(cl(S))$ and (b) holds.

(b)⇒(c): By Lemma 1.1(b) and condition (b), $g^{\#}(int(S)) = (g((int(S))^C))^C = (g(cl(S^C)))^C \subseteq (cl^*(g(S^C)))^C = (cl^*(g^{\#}(S))^C)^C = int^*(g^{\#}(S))$. Hence, $g^{\#}(int(S)) \subseteq int^*(g^{\#}(S))$, and (c) holds.

(c)⇒(d): Let G be any open subset of X, then $G = int(G)$ and (c) implies that $g^{\#}(G) \subseteq int^*(g^{\#}(G))$. Therefore, $g^{\#}(G)$ is σ^*-open set and (d) holds.

(d)⇒(a): Let F be any closed subset of X. Then (d) implies that $g^{\#}(F^C)$ is σ^*-open in Y. Therefore, Lemma 1.1(b) implies that $(g(F))^C = g^{\#}(F^C)$ is σ^*-open in Y. Thus, $g(F)$ is σ^*-closed in Y, and hence g is pointwise-$\mathcal{I}$-closed. □

References

[1] J. Kanicwski and Z.Piotrowski, "Concerning continuity apart from a meager set", *Proc. Amer. Math. Soc.*, **98**(2), 1986, pp. 324-328.

[2] K.Kuratowski, *Topology*, volume I, Academic Press,New York, 1966.

[3] N.S. Noorie and R. Bala, "Some Characterizations of open, closed and Continuous Mappings", *Int. J. Math. Mathematical Sci.*, Article ID527106, 5 pages(2008).

[4] R. Vaidyanathswamy, "The localisation Theory in Set Topology", *Proc. Indian Acad. Sci.*, **20**, 1945, pp. 51-61.

[5] ______, *Set Topology*, Chelsea Publishing Company, New York, 1946.