Applications of Conformal Mapping to Complex Velocity Potential of the Flow of an Ideal Fluid

Mohammed Mukhtar Mohammed Zabih[1], R.M. Lahurikar[2]
[1]Maulana Azad College of Arts, Commerce and Science, Rouza bagh, Aurangabad.431001
[2]Govt. College of Arts and Science, Kile Ark, Aurangabad 431001. (M.S.) INDIA

Abstract-*In this paper we have discussed, an application of conformal mapping to the problems of finding complex velocity potential function $\Omega(z)$ for an irrotational flow of an incompressible fluid, that is, the flow of an ideal fluid in a domain D of the z-plane. In this application, our idea is to device an analytic mapping (in fact conformal mapping) from the z-plane to the w-plane, which maps the domain D conformally on to the domain D' (precisely, either horizontal strip or vertical strip) in the w-plane, where the solution of problem is easy to find. The advantage of this technique is that, the theory of conformal mapping can be employed to reduce a problem to a simpler one whose solution is known. Determining the velocity potential $\Phi(u,v)$ in the w-plane and sending back to $\phi(x,y)$ in the z-plane, gives the complex velocity potential $\Omega(z)= \phi + i\psi$, where ψ is a stream function. This technique is tested through examples.*

Keywords – *complex velocity potential, conformal mapping, Laplace equation, ideal fluid, analytic function.*

1. INTRODUCTION

There are many classes of problemsin mathematics that are difficult to solve in their original form and in the given domain. Conformal mapping, maps an equation and a domain from its original form into another, after some mathematical manipulations we get the solution and the solution is then mapped back to the original form.

If $f(z)= u + i\, v$ is an analytic function in a domain D, then the functions u and v are harmonic [3,4,8] that is, $\nabla^2 u=0$ and $\nabla^2 v=0$ in D [7]. Thus, there is a close connection between analytic functions and Laplace equation. In mathematics, often we want to solve Laplace equation $\nabla^2\phi=0$ in a domain D in z-plane and for the reason of dependence of ϕ on the shape of D, it simply may not be possible to evaluate ϕ. But it may be possible to determine an analytic mapping $f(z)$ from the z-plane to the w-plane so that D', the image of a domain D under $f(z)$, not only has a convenient shape but the function ϕ that satisfy the equation $\nabla^2\phi=0$ in D also satisfies in D' and then return to the z-plane and $\phi(x, y)$ by means of analytic mapping $f(z)$.

If the conformal mapping w= $f(z)$ takes the function $\phi(x, y)$ into a function$\Phi(u,v)$ and if $u + i\, v = f(x + iy)$, where $w = u + iv$ and $z= x + iy$ then

$$\nabla^2 \Phi(u,v)= \left|\frac{dz}{dw}\right|^2 \nabla^2\phi(x, y). \qquad (1.1)$$

So that if $\left|\frac{dz}{dw}\right| \neq \infty$, and if $\nabla^2\phi(x, y)=0$ then by (1.1)$\nabla^2 \Phi(u,v)=0$. Thus, if the function $\phi(x, y)$ is harmonic in the domain D of the z-plane, then the function $\Phi(u,v)$is harmonic in the domain D' of the w-plane. Moreover, any boundary of D in the z-plane along which the function $\phi(x, y)$ is constant is mapped on to a boundary of D' in the w-plane along which the function $\Phi(u,v)$ is constant.

If $F(x, y)$ is the velocity field of planar flow of an incompressible fluid then divF=0 and if the flow is irrotational then curlF=0 [1,2,6] also, there exists a real – valued function $\phi(x, y)$ such that $F(x, y) = \nabla\phi(x, y)$. Thus, divF= $\nabla^2\phi=0$, that is $\phi(x, y)$ satisfies the Laplace equation.

The paper is organized as follows. In section 2, we give the preliminaries [8] and in section 3 we illustrate the applications of the above mentionedtechnique by solving few examples.

2. PRELIMINARIES

Definition 2.1 Conformal Mapping

Let $w = f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D. Then we say that $w = f(z)$ is **conformal** at z_0 if for every pair of smooth oriented curves $C_1$and C_2in D intersecting at $z_0$the angle between $C_1$and C_2 at z_0 is equal to the angle between the image curves $C'_1$and C'_2 at $f(z_0)$in both magnitude and sense[8].

Definition 2.2 Complex Velocity Potential

If F(x, y) is a two dimensional flow of an ideal fluid then there exist a real-valued function $\phi(x, y)$ such that

F$(x, y) = \nabla\phi(x, y)$ (2.1)

The function $\phi(x, y)$ is called velocity potential, which satisfies the Laplace equation $\nabla^2\phi(x, y)$=0, hence is harmonic. The harmonic conjugate $\psi(x, y)$ of velocity potential $\phi(x, y)$ is called the stream function. And the function Ω(z)= ϕ (x, y) + i$\psi(x, y)$ is called the complex velocity potential [8].

Theorem 2.1

If f is an analytic function in a domain D containing z_0, and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0.[8]

Theorem 2.2

Let $f(z) = u(x, y) + iv\,(x, y)$ be an analytic mapping of a domain D in the z-plane onto a domain D' in the w-plane. If the function $\Phi(u,v)$ is harmonic in D', then the function$\phi(x, y) = \Phi(u(x, y), v\,(x, y))$ is harmonic in D[8].

3. EXAMPLES

Example 1. Determine the complex velocity potential Ω(z) of moving ideal fluid in the domain D between two parabolas $y^2 = 4(x + 1)$ and $y^2 = 16(x + 2)$.

Such that

$\phi(x, y) = 10$, if $y^2 = 4(x + 1)$ $= 20$, if $y^2 = 16(x + 4)$ (3.1)

Suppose F(x, y) is the two dimensional velocity field of an ideal fluid, then by equation (2.1)

F$(x, y) = \nabla\phi(x, y)$

Also,$\nabla^2\phi(x, y)$=0 in a domain D satisfying the boundary conditions (3.1)

To determine $\phi(x, y)$ we transform the domain D onto the horizontal strip by a conformal mapping f defined from the z-plane to the upper half of the w-plane, by

$$w = f(z) = z^{\frac{1}{2}} \Rightarrow w^2 = z$$

If $w = u + iv$ and z= $x + iy$ then $x = u^2 - v^2$, y =2 uv. Thus, $y^2 = 4v^2(x + v^2)$

Clearly Re$f(z) = u = \sqrt{\frac{|z|+x}{2}}$ and Im $f(z) = v = \sqrt{\frac{|z|-x}{2}}$ (3.2)

Therefore, the lines v=1 and v =2 in the w-plane corresponds to the parabolas $y^2 = 4(x + 1)$ and $y^2 = 16(x + 2)$in the z-plane (function $w = f(z)$ being one-to-one). Hence the domain D is mapped onto the strip $1 < v < 2$.

Thus, if $w = f(z)$ takes $\phi(x, y)$ into$\Phi(u,v)$ then the transformed boundary conditions are

$\Phi(u,v)$ =10 , if v=1

=20 , if v=2 (3.3)

The shape of strip $1 < v < 2$, $-\infty < u < \infty$, itself suggest that $\Phi(u,v)$ is independent of u. and the transformed Laplace equation $\nabla^2 \Phi(u,v)$=0, reduces to $\frac{\partial^2\Phi}{\partial v^2} = 0$. Therefore, the solution subject to (3.3) is given by $\Phi(u,v)$=10 v.

Thus, the solution of original Laplace equation $\nabla^2\phi(x, y)$=0, by theorem (2.2)and using (3.2) is

$\phi(x, y) = 10\sqrt{\frac{|z|-x}{2}}$

To determine complex velocity potential $\Omega(z)$ of which $\phi(x, y)$ is a real part. We proceed as under.

As $f(z) = u + iv$;$-10\, i\, f(z) = -10\, i\,(u + iv) = 10v - 10iu = \phi\,(x, y)$ + i$\psi(x, y)$, where $\psi(x, y) = -10iu = -10i\sqrt{\frac{|z|+x}{2}}$

Since,$f(z)$is analytic in D, it follows that the function $-10if(z)$ is also analytic in D.

Thus, $\Omega(z) = -10if(z) = -10\, iz^{\frac{1}{2}}$ is the desired complex velocity potential.

Example **2.** Determine the complex velocitypotential $\Omega(z)$ of moving ideal fluid in the domain D between two circles $|z - i| = 1$ and$|z - 2i| = 2$

Such that
ϕ(x,y) =20 , if $x^2 + (y-2)^2 = 4$
= 40 , if $x^2 + (y-1)^2 = 1$ (3.4)
Suppose F(x, y) is the two dimensional velocity field of an ideal fluid, then by equation (2.1)

F$(x, y) = \nabla\phi(x, y)$

Also $\nabla^2\phi(x, y)$=0 in a domain D satisfying the boundary conditions (3.4)

To determine $\phi(x, y)$ we transform the domain D onto horizontal strip by a conformal mapping $f$defined from the z-plane to the w-plane, by

$$w = f(z) = \frac{1}{z}$$

Clearly,$f(z)$is one-to-one conformal mapping from the z-plane to the w-plane

If w= $u + iv$ and z= $x + iy$ then $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

Clearly Re$f(z) = u = \frac{x}{x^2+y^2}$ and Im $f(z) = v = \frac{-y}{x^2+y^2}$ (3.5)

To find the image of the boundaries consider

$$|z - i| = 1 \Rightarrow x^2 + y^2 - 2y = 0$$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{2v}{u^2+v^2} = 0$$

$$\Rightarrow 1+2\,v = 0 \Rightarrow v = \frac{-1}{2}$$

Also,$|z - 2i| = 2 \Rightarrow x^2 + y^2 - 4y = 0$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

$\Rightarrow$ 1+4 v =0 $\Rightarrow$ $v = \frac{-1}{4}$

Thus, the boundaries $|z - i| = 1$ and $|z - 2i| = 2$ of the domain D are mapped onto the lines

$v = \frac{-1}{2}$ *and* $v = \frac{-1}{4}$ respectively of the horizontal strip $\frac{-1}{2} < v < \frac{-1}{4}$. Thus, the domain D is mapped onto the horizontal strip $\frac{-1}{2} < v < \frac{-1}{4}$.

Thus, if $w = f(z)$ takes $\phi(x, y)$ into $\Phi(u,v)$ then the transformed boundary conditions are

$\Phi(u,v) = 20$, if $v=\frac{-1}{4}$

$=40$, if $v=\frac{-1}{2}$ (3.6)

As in example 1 the solution of transformed equation, $\nabla^2 \Phi(u,v)=0$, subject to (3.6) is given by $\Phi(u,v)=-80\,v$.

Thus, the solution of original Laplace equation, $\nabla^2\phi(x, y)=0$ using (3.5) is

$$\phi(x, y) = 80\,\frac{y}{x^2 + y^2}$$

To determine complex velocity potential $\Omega(z)$ of which $\phi(x, y)$ is a real part. We proceed as under.

As $f(z) = u + iv$; $80\,i\,f(z) = 80\,i\,(u + iv) = -80v + 80iu = 80\,\frac{y}{x^2+y^2} + 80i\,\frac{x}{x^2+y^2} = \phi(x, y) + i\psi(x, y)$, where $\psi(x, y) = 80iu = 80i\,\frac{x}{x^2+y^2}$

Since , $f(z)$ is analytic in D, it follows that the function 80 $if(z)$ is also analytic in D.

Thus, $\Omega(z) = 80\,if(z) = 80i\frac{1}{z}$ is the desired complex velocity potential.

Example 3. Determine the complex velocity potential Ω(z) of moving ideal fluid in the domain D, the wedge between two rays, $arg\, z = \frac{-\pi}{4}$ and $arg\, z = \frac{\pi}{4}$

Such that

$\phi(x, y) = 0$, if $arg\, z = \frac{-\pi}{4}$

$=30$, if $arg\, z = \frac{\pi}{4}$ (3.7)

Suppose F(x, y) is the two dimensional velocity field of an ideal fluid, then by equation (2.1)

$F(x, y) = \nabla\phi(x, y)$

Also $\nabla^2\phi(x, y)=0$ in a domain D satisfying the boundary conditions (3.7)

To determine $\phi(x, y)$ we transform the domain D onto horizontal strip by a conformal mapping f defined from the z-plane to the w-plane, defined by the principal branch of the complex logarithm
$w = f(z) = Ln\, z = log_e r + i\theta, -\pi < \theta \le \pi\, , r > 0$
Where $r = |z|$ and $\theta = \arg z$.
If $w = u + iv$ and $z = x + iy$ then $u = log_e r, v = \theta$

The function being one-to-one, the lines $v = \frac{-\pi}{4}$ and $v = \frac{\pi}{4}$ corresponds to the rays,

$arg\, z = \frac{-\pi}{4}$ and $arg\, z = \frac{\pi}{4}$ emanating from origin (Note that origin is not included in the domain). Therefore, the domain D is conformally mapped on to the horizontal strip $\frac{-\pi}{4} < v < \frac{\pi}{4}$.

Thus, if $w = f(z)$ takes $\phi(x, y)$ into $\Phi(u,v)$ then the transformed boundary conditions are

$\Phi(u,v) = 0$, if $v = \frac{-\pi}{4}$

$= 30$, if $v = \frac{\pi}{4}$ (3.8)

Therefore, the solution subject to (3.8) is given by $\Phi(u,v)=\frac{60}{\pi}v + 15$

Thus, the solution of original Laplace equation $\nabla^2\phi(x, y)=0$ is

$\phi(x, y) = \frac{60}{\pi}tan^{-1}\frac{y}{x} + 15$

To determine complex velocity potential $\Omega(z)$ of which $\phi(x, y)$ is a real part. We proceed as under.

As $f(z) = u + iv; -60\,\frac{i}{\pi}\,f(z) = -60\,\frac{i}{\pi}(u + iv) = \frac{60}{\pi}\,v - \frac{60}{\pi}iu + 15 = \phi\ (x, y) + i\psi(x, y)$, where $\psi(x, y) = -\frac{60}{\pi}iu = \frac{60}{\pi}i\ log_e r$

Since, $f(z)$is analytic in D, it follows that the function $-60\frac{i}{\pi}f(z)$ is also analytic in D.

Thus, $\Omega(z) = -60\frac{i}{\pi}f(z) = -60\frac{i}{\pi}Ln\ z$ is the desired complex velocity potential.

Example 4. Determine the complex velocity potential Ω(z) of moving ideal fluid in the domain D between two hyperbolas$4x^2 - 12y^2 = 3$ and $12x^2 - 4y^2 = 3$,

Such that

$\phi(x, y) = 5$, if $4x^2 - 12y^2 = 3$

$= -5$, if $12x^2 - 4y^2 = 3$ (3.9)

SinceF$(x, y) = \nabla\phi(x, y)$ and $\nabla^2\phi(x, y)=0$ in a domain D satisfying the boundary conditions (3.9)

Let f be defined from the z-plane to the domain$\frac{-\pi}{2} < u < \frac{\pi}{2}$and $-\infty < v < \infty$, a vertical strip, defined as $w = f(z) = sin^{-1}z$, $-1 < Rez < 1$ and $-\infty < Imz < \infty$.

If $w= u + iv$ and $z= x + iy$ then $x = sin\,u\ cos\,h\,v$, $y = cos\,u\ sin\,h\,v$.

Thus, $$\frac{x^2}{sin^2u} - \frac{y^2}{cos^2u} = 1$$

Therefore, the lines $u =\frac{\pi}{3}$and$u =\frac{\pi}{6}$in the w-plane corresponds to the hyperbolas $4x^2 - 12y^2 = 3$ $and 12x^2 - 4y^2 = 3$ in the z-plane (function $w = f(z)$ being one-to-one). Hence the domain D is mapped onto the vertical strip$\frac{-\pi}{2} < u < \frac{\pi}{2}$and
$-\infty < v < \infty$.
Thus, if $w = f(z)$ takes $\phi(x, y)$ into$\Phi(u,v)$ then the transformed boundary conditions are

$\Phi(u,v) = 5$, if $u =\frac{\pi}{3}$

$= -5$, if $u =\frac{\pi}{6}$ (3.10)

In the vertical strip$< u < \frac{\pi}{2}$, $-\infty < v < \infty$, $\Phi(u,v)$ is independent of v and the transformed Laplace equation $\nabla^2\ \Phi(u,v)=0$, reduces to $\frac{\partial^2\Phi}{\partial u^2} = 0$. Therefore, the solution subject to (3.10) is given by $\Phi(u,v)=\frac{60}{\pi}\ u - 15$.

Thus, the solution of original Laplace equation $\nabla^2\phi(x, y)=0$

$$\phi(x, y) = \frac{60}{\pi}\ u - 15$$

as $w = f(z) = sin^{-1}z$, by Taylor's series expansion $w = z + \frac{1}{2}\frac{z^3}{3} + \frac{1}{2}\frac{3}{4}\frac{z^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{z^7}{7} + \ldots$

$u + iv = x + iy + \frac{1}{2}\frac{(x+iy)^3}{3} + \frac{1}{2}\frac{3}{4}\frac{(x+iy)^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{(x+iy)^7}{7} + \ldots$

therefore $u = x + \frac{1}{2}\frac{1}{3}(x^3 - 3xy^3) + \frac{1}{2}\frac{3}{4}\frac{1}{5}(x^5 - 10x^3y^2 + 5xy^4) + \cdots$

and $v = y + \frac{1}{2}\frac{1}{3}(3x^2y - y^3) + \frac{1}{2}\frac{3}{4}\frac{1}{5}(5x^4y - 10x^3y^3 + y^5) + \cdots$

To determine complex velocity potential $\Omega(z)$ of which $\phi(x, y)$ is a real part. We proceed *as* under.

As $f(z) = u + iv$; $\frac{60}{\pi} f(z) - 15 = \frac{60}{\pi}(u + iv) - 15 = \frac{60}{\pi}u + \frac{60}{\pi}iv - 15 = \frac{60}{\pi}u - 15 + \frac{60}{\pi}iv = \phi(x, y) + i\psi(x, y)$, where $\psi(x, y) = \frac{60}{\pi}iv$

Since, $f(z)$is analytic in D, it follows that the function $\frac{60}{\pi}iv\, f(z) - 15$ is also analytic in D.

Thus, $\Omega(z) = \frac{60}{\pi}iv\, f(z) - 15 = \frac{60}{\pi}iv\, sin^{-1}z - 15$ is the desired complex velocity potential.

Figures: The domain and codomain for example 1, example 2, example 3 and example 4 are shown in figure 1, figure 2, figure 3 and figure 4 respectively.

Fig. 1

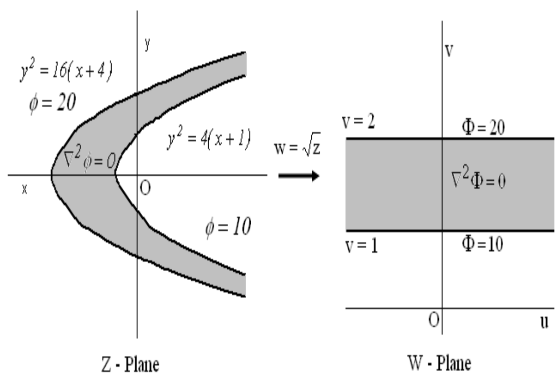


Fig.2

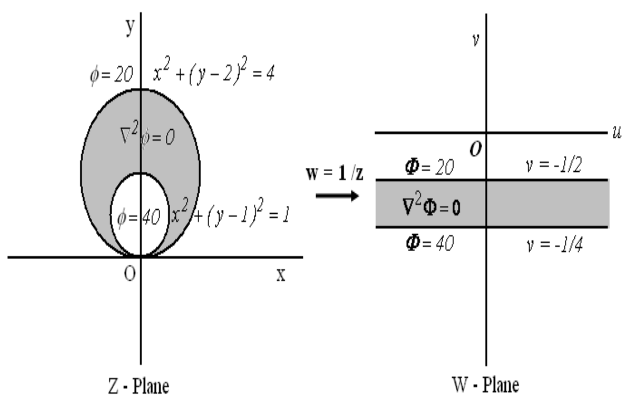


Fig.3

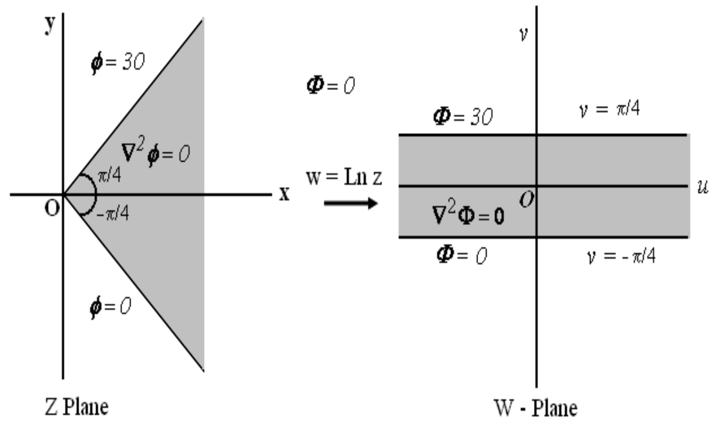
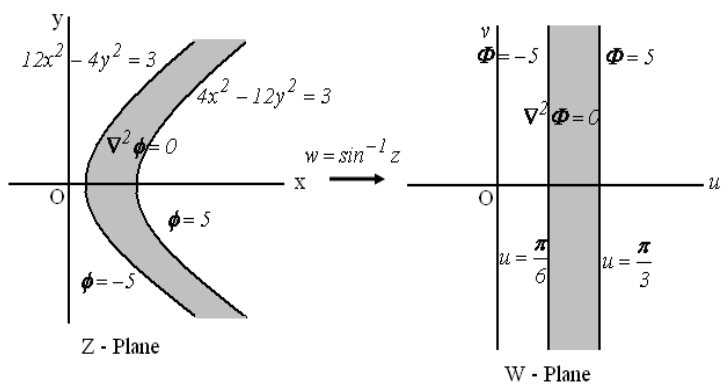


Fig.4



Conclusion:

Complex velocity potential of an ideal fluid can be determined by solving problem in either horizontal strip or vertical strip, which has simple geometric shape than any other type of region.

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